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THE
THEORY OF PLANE CURVES

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BY

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In

Memory of

SIR ASUTOSH MOOKERJEE

PREFACE TO THE SECOND EDITION

The present work is an endeavour to give as complete an account of the properties of cubic and quartic curves as could be compressed within the limits of a single volume of moderate size. It is a revised edition of my Lectures on the Theory of Plane Curves, Part II, which was published sometime ago, the revision having been undertaken with the object of bringing the contents of the previous work up to date as far as practicable. The necessity for the revision arose partly from the advancing requirements of the class of students for which the book was originally written and partly from the suggestions of teachers that its scope and usefulness might be extended by incorporating materials necessary for a systematic study of the subject. The entire book has accordingly been rewritten, new materials incorporated and the articles re-numbered and the present work is published as a separate companion volume to my Theory of Plane Curves, Vol. I, applying in its construction the theory developed in that book to the study of the curves of certain orders, particularly cubics and quartics. The subject covers so extensive a field that a word is necessary to explain the plan adopted, which, I trust, will facilitate to some extent the study of a variety of algebraic and other curves.

In the first place, the subject has been presented in clear and concise form to students commencing a systematic study of the higher curves, indicating at the same time, references, as far as practicable, to original sources in order to enable the senior student to consult them. Secondly, a

large number of examples have been given by way of illustration and as exercises for practice. Analysis has not been exclusively used, and where simplicity could be gained, geometrical processes have not been neglected.

Properties of cubic curves have been studied in Chapters I to III with special reference to their mode of generation, singularities, inflexions, harmonic polars, etc., and for this purpose canonical forms, which have often been found useful, are dealt with in Chapter II. The theory of elliptic integrals has been applied to discuss certain properties of non-singular cubics in Chapter III. But a fuller treatment was neither possible nor desirable within the narrow limits of the present work. As rightly remarked by Salmon "such a treatment in order to have any pretensions to completeness ought to contain an account of the application, made by Clebsch in continuation of Riemann's researches on elliptic and Abelian integrals, to the theory of curves." Chapter IV deals with some *special* cubics of historic importance, care being taken not to embarrass the student with materials of a complex nature. Only general characteristics have been dealt with in the hope that they will afford sufficient materials for independent thinking in more advanced stages and in more complicated cases. Chapter V treats of Invariants and Covariants of cubic curves, but no attempt has been made to give a complete system of invariants and covariants to classify cubic curves by means of them, nor has the interesting subject of Apolarity been dealt with.

The subject of quartic curves is too extensive to be adequately considered in a small work like this. The discussion has, therefore, been chiefly confined in Chapters VI and VII, to the prominent characteristics of these

curves, with special reference to their mode of generation, singularities, bitangents, etc. Chapter VIII has been entirely allotted to unicursal quartics and the whole of Chapter IX to the consideration of binodal and bicircular quartics. Circular cubics have also been studied in Chapter X as *degenerate* bicircular quartics. In Chapter XI some well-known curves of order four possessing interesting properties have been discussed. The last Chapter is taken up with the discussion of Roulettes, Cycloids, Spirals, etc.

Appendix I contains a note on the sextactic points on cubics, based on Cayley's well-known paper—On the conic of five-pointic contact at any point of a plane curve [Phil. Trans., Vol. CXLIX (1859). Vol. IV, pages 207-39 of the collected works]. But it may be added that sextactic points are best dealt with by means of the famous Theorem of Abel in Elliptic Integrals, as related to points on a plane cubic. Appendix II gives some properties of bicircular quartics based on Prof. Casey's interesting paper published in the Transactions, Vol. 24, of the Royal Irish Academy (1869).

In the preparation of the book I have freely consulted the works of Salmon, Clebsch, Wielitner, Basset, Pascal, Hilton, Loria, Teixeira, etc., and the several papers of Zeuthen and Noether, published in journals and periodicals. References were traced mainly from Pascal's Repertorium and Prof. Hilton's Plane Algebraic Curves was very useful in the selection of appropriate examples for illustration. Much interesting and important information with regard to special curves was gathered from Loria's and Teixeira's books. My indebtedness to these authors is hereby gratefully acknowledged, and it is impossible to record in detail how much help I have derived from their classical works.

In concluding this preface, I desire to say that in addition to the works of authors mentioned above, I have consulted with much advantage some notes on cubics and quartics by my colleague Dr. Haridas Bagchi, Ph.D., and I am indebted for useful hints and suggestions to Mr. A. C. Bose, M.A., Controller of Examinations, University of Calcutta. Once more I register here my deep indebtedness to the late lamented Sir Asutosh Mookerjee, Kt., but for whose kindness in encouraging me to revise the original lecture notes for the press and his subsequent suggestion for a *revised* second edition, the present work could not have been an accomplished fact. The volume is dedicated to his sacred and revered memory.

I need hardly say that I shall be very grateful for any corrections or suggestions for the improvement of the work which I may receive from teachers or students who may use it.

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CHAPTER I

Curves of the Third Order

Section I. General Properties.

1. The General Equation of Cubic Curves:

The general equation of a curve of the third order involves ten arbitrary constants, and consequently, the number of disposable constants in the equation is *nine*, and a cubic can be made to pass through any nine points.

The general equation of a cubic curve in homogeneous co-ordinates, arranged in powers of x, can be written in the symbolic form as—

$$u_0 z^3 + u_1 z^2 + u_2 z + u_3 = 0$$

where u_0 is a constant and u_1, u_2, u_3 are homogeneous expressions of the first, second and third orders respectively in x and y. The equation contains nine disposable constants and therefore determines a general curve of the third order.

2. Classification of Cubic Curves:

In the classification of curves those distinctive properties are regarded as fundamental which are unaffected by projection, and for purposes of classifying cubic curves, the properties of the configuration of the inflexions and the constant cross-ratio properties* of any four tangents drawn from any point on the cubic have generally been used. Newton,† basing his classification on the nature of the infinite branches, divided the cubics into fourteen genera,

* Salmon, *Théorèms sur les courbes du troisième degré*, Crelle, Bd. 42 (1851).

† Newton, *Enumeratio linearum tertii ordinis* (1706).

containing 72 species to which six others were added, four by Stirling and two by Cramer, making in all 78 species. Plücker * likewise established 61 divisions called *groups*, and based his classification of cubics on the nature of the points at infinity, and obtained 219 different species. Prof. Cayley † re-examined this division into groups and compared them with Newton's species. Salmon ‡ gave as many as thirty species and his classification proceeds almost on the lines of Newton. Möbius, § however, investigated a projective classification of cubics, based on the properties of cubic cones, and obtained 7 classes of curves. Finally, Wiener || gave 13 different species of cubic curves, by using a canonical form,

$$x^3+y^3+z^3+6\lambda xyz=0$$

of which ten are non-singular and three with double points.

3. Newton's Method of Classification:

Newton's method ¶ of classification of cubic curves depends upon reducing the equation of the curve to the form—

$$y^2=ax^3+bx^2+cx+d$$

or

$$y^2=a(x-\alpha)(x-\beta)(x-\gamma) \qquad \cdots \quad (1)$$

where α, β, γ are the roots of the right-hand side expression equated to zero. Newton calls this a *divergent parabola*, and established the theorem that every cubic may be

* Plücker, *System der Analytische Geometrie* (1835).

† Cayley, *On the Classification of Cubic Curves*. Transactions of the Cambridge Philosophical Society, Vol. II (1866), Part I, pp. 81-128, Coll. Works, Vol. V, pp. 354-399.

‡ Salmon, H. P. Curves §§ 202-209.

§ Möbius, Leipzig Abhandlungen, Vol. I (1852), p. I.

|| Wiener, *Die einleitung der eb. Kurven*, etc. (1901).

¶ Rouse Ball, *On Newton Classification of Cubic Curves*, Proc. Lond. Math. Soc., Vol. 22 (1891), p. 140.

projected into one of the five divergent parabolas given by the equation (1), according to the nature of α, β, γ. Since a real cubic has at least one real point of inflexion, the transformation can be effected by taking for z the tangent at the real inflexion, and for y the harmonic polar, and then projecting the inflexional tangent to infinity, so that the harmonic polar now becomes a diameter of the cubic.

Case I. When α, β, γ are all real and distinct. The curve consists of an oval and an infinite branch, and is called by Newton *Parabola companiformis cum ovali*. Cayley calls it a *complex*.*

Case II. If α is real, but β and γ conjugate imaginaries:

The equation can be written in the form—

$$y^2=a(x-\alpha)\ \{(x-p)^2+q^2\}$$

It is seen that real points will be obtained only when $x\geq\alpha$, whence it follows that the curve has only an infinite branch, and is called *Parabola pura* (Cayley's *simplex*).

The classes (I) and (II) have no singular points and are the typical fundamental forms of all non-singular curves of the third order, whose discriminant does not vanish.

If the curve is given in any system of co-ordinates, to determine its nature we have to ascertain whether it can be brought to the above two forms; and if the discriminant is positive, it is a *complex*, and if negative, a *simplex*.†

Case III. If $\alpha=\beta$ and γ real, the equation takes the form

$$y^2=a(x-\alpha)^2(x-\gamma)$$

and shows that when $\alpha<\gamma$, the oval shrinks into an isolated point (acnode), and the curve consists of an infinite branch

* Cayley, *On the inflexions of the cubical divergent parabolas*—Quart. Journal of Math., Vol. 6 (1864), or Coll. Works, Vol. V, pp. 284-288.

† Cremona, *Considerazioni sulle curve piane del terz' ordine*—Giorn di Mat., Vol. 2(1) (1864), p. 78.

and a conjugate point, and is called *Parabola punctata* (Cayley's *acnodal* cubic).

Case IV. When a is real and $\beta=\gamma$. Then from the equation $y^2=a(x-\alpha)(x-\beta)^2$ it is seen that $(\beta, 0)$ is a double point, and real points are obtained for $x>\alpha$. The curve is called the *Parabola nodata* (Cayley's *crunodal* cubic).

Case V. If α, β, γ are all real and equal, the equation becomes

$$y^2=a(x-\alpha)^3$$

The curve possesses a cusp and is called *Parabola cuspidata* (Cayley's *cuspidal* cubic).

4. CHASLES' CLASSIFICATION :

Chasles * established a supplement of Newton's theorem namely, *every curve of the third order can be projected into one of five central cubics*, given by—

$$y=ax^3+3bx^2y+3cxy^2+dy^3$$
$$=a(x-\alpha y)(x-\beta y)(x-\gamma y) \qquad \dots \quad (2)$$

This form is obtained by projecting the harmonic polar, instead of the inflexional tangent, to infinity. The inflexion then becomes a centre and every chord through it is bisected. Proceeding as before, it is found that the corresponding curve belongs to one of five different categories.† It is not considered necessary to enter into further details of the theory in the present work, but the reader is referred to G. Loria—*Spezielle Algebraische und Transzendente ebene Kurven*, Vol. I (1910), pp. 14-25, where a complete list of references and enumeration of the different species of Newton are to be found.

* Chasles, Aperçu historique (Bruxelles, 1837), Note XX.

† J. G. Teixeira calls them "Chasles' cubics"—Tratado de las curvas especiales notables (Madrid, 1905).

5. Classitication based on Deficiency:

A curve of the third order can have at most one double point, and no other multiple point. Hence, according to their deficiencies, cubic curves may be divided into the two following fundamental classes:

(1) *Non-singular* or *anautotomic cubics*, which have no double points, and the deficiency is unity.

(2) *Singular* or *autotomic cubics*, which have a double point, and the deficiency is zero.

Singular cubics can again be subdivided into three species according to the nature of the double point:

(1) *Nodal cubics*—in which the double point is a node, with two distinct tangents (real).

(2) *Cuspidal cubics*—in which the double point is a cusp with coincident tangents.

(3) *Acnodal cubics*—in which the double point is a conjugate point with imaginary tangents.

The characteristics of these three classes of cubics, calculated by means of Plücker's formulæ, are *:—

	n	δ	κ	m	τ	ι	p
Non-singular	3	0	0	6	0	9	1
Nodal or Acnodal	3	1	0	4	0	3	0
Cuspidal ...	3	0	1	3	0	1	0

* Theory of Plane Curves, Vol. I, § 146.

6. Particular Forms of Equations:

The homogeneous equation of a cubic passing through the vertex A of the triangle of reference is—

$$x^2u_1+xu_2+u_3=0 \qquad \ldots \quad (3)$$

where u_1, u_2, u_3, are functions of y, z, of orders 1, 2 and 3, and $u_1=0$ represents the tangent at the vertex.

If, however, the vertex A is a double point on the curve, the equation should contain no x^2, and consequently, it takes the form $xu_2+u_3=0$, where u_2 and u_3 are homogeneous functions of the second and third orders respectively in y and z, and $u_2=0$ is the equation of the tangents at A. Hence, the point A will be a node, a cusp, or a conjugate point on the curve, according as u_2 represents two *real and distinct*, *coincident*, or *imaginary* right lines.

If the vertex A is a point of inflexion on the curve, the tangent at A meets it in three consecutive points. Hence, if y be eliminated between $u_1=0$ and the equation (3), the resulting equation should have z^3 as a factor, which requires that the co-efficient of x should vanish, *i.e.*, u_2 should contain u_1 as a factor. Thus, the equation of a cubic having a point of inflexion at A is—

$$x^2u_1+xu_1v_1+u_3=0 \qquad \ldots \quad (4)$$

The equation of a cubic circumscribing the triangle of reference is of the form

$$x^2u+y^2v+z^2w+kxyz=0 \qquad \ldots \quad (5)$$

where u, v, w are linear expressions in y, z; z, x and x, y respectively.

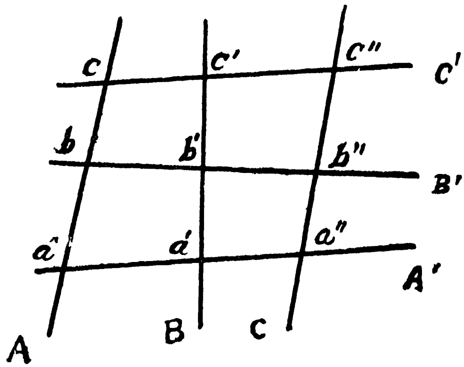
7. Cubics through Nine Points:

Let $u=0$ and $v=0$ be the equations of any two cubic curves. Then, $u=kv$ represents a cubic curve passing through the nine intersections of the curves u and v.

If, however, u and v are degenerate, consisting each of three right lines, the equation of a cubic may be written in the form

$$uvw = ku'v'w' \qquad \ldots \quad (6)$$

This shows that the cubic passes through all the nine intersections of the straight lines u, v, w, u', v', w', *i.e.*, through the points $a, b, c, a', b', c', a'', b'', c''$, as in the figure.



That the equation of all cubics can be put into the form (6) can be shown as follows:—

The equation of a cubic circumscribing the triangle of reference is of the form—

$$x^2u_1 + xu_2 + yz(my + nz)$$
$$\equiv x(xu_1 + u_2 - lyz) + yz(lx + my + nz) = 0$$

where l is at our disposal. If now we determine l so that the discriminant of $xu_1 + u_2 - lyz$ vanishes, the equation of the cubic reduces to the desired form.

We may state the following theorem:

If the three points of intersection of a line with a cubic are joined to those of another line, the joining lines meet the cubic again in three collinear points.

8. THE TANGENTIAL OF A POINT:

DEFINITION: The point where the tangent at a given point on a cubic meets the curve again is called the *tangential* * of the point of contact.

If in the equation (6) we put $u=v$, *i.e.*, if the lines A and B coincide, the equation takes the form

$$u^2w=k\,u'v'w' \quad \ldots \qquad \ldots \quad (7)$$

which shows that u', v', w' are tangents, the points of contact lying on the line u.

The same equation further shows that the three other points where these tangents meet the cubic again lie on the line w.

Hence we obtain the theorem †:

If a right line meets a cubic in three points, the tangents at those points meet the cubic again in three other collinear points, or in other words, *the tangentials of three collinear points are collinear.*

It is to be noted that if the tangential coincides with the point, the tangent has a contact of the second order, and the point is an inflexion on the cubic.

9. THE SATELLITE OF A RIGHT LINE:

DEFINITION: The line on which lie the tangentials of three collinear points is called the *satellite* of the line of collinearity.

Thus, in the preceding article, the line w is the satellite of the line u.

* This point is also called the *satellite point* of the tangent—Cayley, Coll. Works, Vol. II, p. 409.

† This theorem is due to Maclaurin. See Jonquières, *Mélanges de géometric* (1856), p. 223.

If in the equation (7), we suppose the line u to move to infinity, the lines u',v',w' become tangents whose points of contact are at infinity, *i.e.*, the lines u',v',w' are asymptotes of the cubic. We thus obtain the theorem*:

The three points in which a cubic intersects its asymptotes lie on a right line, which is called the satellite of the line at infinity.

The equation (7) now takes the form—

$$I^2 w = ku'v'w' \quad \ldots \qquad \ldots \quad (8)$$

where $I=0$ is the line at infinity. Remembering that the quantities u',v',w' and w are proportional to the lengths of the perpendiculars drawn from a point on the curve to those lines, the equation (8) admits of the following geometrical interpretation:

The product of the perpendiculars drawn from any point on the cubic to its asymptotes is in a constant ratio to the perpendicular drawn from the same point to the satellite of the line at infinity.

10. THEOREM:

If two of the points of intersection of a line with a cubic be points of inflexion, the third must also be a point of inflexion.

The tangent at a point of inflexion has three-pointic contact with the curve. Consequently the tangential of a point of inflexion coincides with the point itself. Thus, if the points a and b are points of inflexion, their tangentials respectively coincide with them, *i.e.*, the satellite coincides with the line ab itself. Hence, the tangential of the third point c, where ab meets the cubic, coincides with c, *i.e.*, c is a point of inflexion.

* Poncelet, Crelle, Bd. 8 (1832) p. 130.

If in the equation (6), $u'=v'=w'$, the equation takes the form $uvw=ku'^3$, which shows that u, v, w are the inflexional tangents, their points of contact lying on u'. Thus,

If a cubic has three real inflexions, they lie on a right line.

11. THEOREM :

*A curve of the third order cannot have more than three real points of inflexion.**

If the vertices B and C of the triangle of reference are two real points of inflexion on the curve, the third real point of inflexion must lie on the line BC ($x=0$). Hence the equation of the cubic can be put into the form

$$(x+nz)(x+my)(x+\mu y+\nu z)=\lambda x^3 \qquad \dots \quad (9)$$

If, again, the vertex A be a real point of inflexion on the curve, the co-efficient of x^3 should be wanting in the equation (9) and the co-efficient of x should contain that of x^2 as a factor, which requires that $\lambda=1$, and

$$(m+\mu)(n+\nu)(\mu n+\nu m+mn)$$
$$=(n+\nu)^2\mu m+(m+\mu)^2 n\nu \qquad \dots \quad (10)$$

Now, putting $m\alpha=\mu$ and $n\beta=\nu$, the second condition reduces to $(1+\alpha)^2+(1+\beta)^2=(1+\alpha)(1+\beta)$, which is a quadratic in $\dfrac{1+\alpha}{1+\beta}$, with complex roots. Hence it is not possible to assign real values to m, n, which will satisfy (10). Therefore, the vertex A cannot be a real point of inflexion.

12. From what has been said in Art. 8, it follows that if tangents be drawn from three collinear points a,b,c, on a cubic, then the line joining the point of contact of *any* one of the tangents from a to that of *any* one of the tangents from b passes through the point of contact of *one* of the tangents from c. But from each of the points a, b, c, four tangents can be drawn to the cubic.

* This proof is taken from Basset's *Cubic and Quartic Curves.*

Consequently, the points of contact of these tangents lie on *sixteen* different lines which have the line *abc* for satellite. Hence, we obtain the theorem*:

A given line has only one satellite, but is itself the satellite of sixteen different lines.

It is to be noted that the twelve points of contact lie on sixteen different lines, three on each, and the sixteen lines pass through the twelve points, four through each.†

13. THEOREM:

The four points of contact of tangents drawn from any point on a cubic are the vertices of a quadrilateral whose diagonal points have the same tangential as the given point. ‡

Let a_1,a_2,a_3,a_4, b_1,b_2,b_3,b_4, and c_1,c_2,c_3,c_4 be the points of contact of tangents drawn from three collinear points A,B,C respectively. These points lie on sixteen lines. Suppose A and B coincide. Then, this coincident point must be one of the four points c_1,c_2,c_3,c_4, say (c_1). The sixteen lines now reduce to the six sides of the quadrilateral $a_1a_2a_3a_4$, each taken twice and the tangents at the vertices a_1,a_2,a_3,a_4. Hence the intersection of a pair of opposite sides is one of the three points c_2,c_3,c_4, and the tangents at the vertices meet the cubic in the same point A, *i.e.*, c_2,c_3,c_4 are the diagonal points of the quadrilateral $a_1a_2a_3a_4$.

Cor. The following theorem is easily deduced:

Two tangents being drawn from any point on a cubic, the tangent at the third point, where the chord of contact meets the curve, meets the tangent at the given point at a point on the curve.

* Dumont, *Introduction à la géométrie des courbes du troisième ordre* (1904), p. 73.

† Hesse, Crelle, Bd. 36 (1848) p. 143.

‡ Cremona, *Introduzioni ad una teoria geometrica delle curve piane* (1861), §146. For other similar properties, see Mangeot, Bull. Soc. Math. de France, t. 21 (1893), p. 44.

Ex. 1. Two cubics with the same asymptotes intersect in three finite collinear points.

Ex. 2. If six of the intersections of two cubics lie on a conic, the other three are collinear.

Ex. 3. Every curve of the *n-th* degree which passes through $3n-1$ fixed points on a cubic passes through one other fixed point on the curve.

Ex. 4. Discuss the case when $n=1$ or 2.

Ex. 5. Three cubics pass through seven points. Prove that the line joining the remaining intersections of the cubics taken in pairs form a triangle whose vertices lie one on each of the cubics. [Apply the principles of residuation.]

14. STEINER'S POLYGON :

Let P and Q be any two fixed points on a cubic. Join a third point A_1 on the cubic with P, and let PA_1 meet the cubic in a third point A_2. Join A_2Q and let A_2Q meet the curve in A_3. Join A_3P and let A_4 be the point where A_3P meets the curve, and so on. In this way we obtain a polygon $A_1A_2A_3A_4$......inscribed in the cubic such that the sides ultimately pass through two fixed fundamental points.

There are then two cases possible :—either the series is not closed, when the construction can be further continued or it is always closed by the same even number $2n$ of sides. The figure thus obtained is called a *Steiner's polygon*, and P and Q are said to form a *Steiner's pair* of order n.*

If P, Q are Steiner's pair of order n, and R is the point where PQ meets the cubic and S the point of contact of one of the tangents from R, then PS and QS are Steiner's pairs of order $2n$.†

If the tangentials of two points are a Steiner's pair of order n, the points themselves form a Steiner's pair of order $2n$.†

* Steiner, Crelle, Bd. 32 (1846), p. 182.

† Clebsch, Crelle, Bd. 63 (1864), p. 94, and Cruber, Crelle, Bd. 114 (1894), p. 312.

If a Steiner's pair is projected from any point of a cubic, the projections form a Steiner's pair of the same order.

The same process can be applied in constructing a polygon inscribed in a binodal quartic, taking those nodes for the fixed fundamental points. Prof. Steiner enunciated these propositions and a number of others in his paper *Geometrische Lehrsätze*—Crelle, Bd. 32 (1846), pp. 182-184, without proof, but these were proved by using elliptic functions by Clebsch—Ueber einen Satz Von Steiner und einige Punkte der Theorie der Curven dritter Ordnung—Crelle, Bd. 63 (1864), pp. 106-121.

They were considered geometrically by P. H. Schoute—Die Steinerschen Polygone—Crelle, Bd. 95 (1883), pp. 105-119. For further information on Steiner's polygon and other polygons connected with a cubic curve, the student is referred to Clebsch's paper cited above, and also the paper by H. S. White—Annals of Mathematics (Cambridge) Vol. 7 (2) (1906) p. 172.

15. THEOREM:

If two pairs of opposite sides of a hexagon inscribed in a cubic meet in points on the cubic, the third pair also intersects on the curve.

Let the first, third and fifth sides of the hexagon be denoted by P, Q, R and the second, fourth and sixth sides by P′, Q′, R′ respectively. Let P, P′ intersect at A, and Q, Q′ at B on the cubic. Then the third pair R, R′ must intersect at a point on the curve.

For, consider the following cubics through eight points, *i.e.*, the six vertices of the hexagon and the two points A, B:—

(1) the given cubic,

(2) the cubic consisting of the lines P, Q, R,

(3) „ „ „ „ „ P′, Q′, R′.

Therefore they must pass through a ninth* common point, *i.e.*, R and R′ intersect the cubic at the same point C on the curve.

Cor. If R and R′ are tangents to the cubic, the theorem becomes:—

If the opposite sides of a quadrilateral inscribed in a cubic intersect on the curve, the tangents at the opposite vertices meet at the same point on the cubic, i.e., the opposite vertices have the same tangential point.

It may be shown further that the tangentials of three pairs of opposite vertices are collinear.†

16. INTERSECTIONS OF A CONIC WITH A CUBIC:

Let a conic meet a cubic in six points A, B, C, D, E, F, and let AB, CD and EF meet the cubic again at the points P, Q, R. Then P, Q, R are collinear.

Consider the three cubics through eight points, *i.e.*, ABCDEFPQ.

(1) The given cubic,

(2) The lines AB, CD, EF,

(3) The conic and the line PQ.

They must then pass through a ninth common point, namely, the point R, which must be on PQ, since it cannot lie on the conic.

If, however, A, C, E, respectively coincide with B, D, F, the conic has simple contact with the cubic at A, C, E, and P, Q, R become the tangentials of these points. Hence we obtain the theorem: ‡

The tangentials of the three points, at which a conic has simple contact with a cubic curve, are collinear.

* Theory of Plane Curves, Vol. I, § 28.

† Cayley, "*Memoire sur les courbes du troisieme ordre,*" Coll. Papers, Vol. I, p. 184.

‡ Durége, *Die ebenen Kurven dritter Ordnung* (1871), p. 136.

Ex. 1. The tangentials of six points of intersection of a conic with a cubic lie on a conic. (Cremona § 145). [Apply residuation].

Ex. 2. A triangle is inscribed in a cubic. If the tangentials of two vertices lie on the opposite sides, shew that the tangential of the third vertex also lies on the opposite side. (Durége).

[Consider the three cubics—the given cubic, the three sides and the three tangents at the vertices. They pass through nine common points.]

17. This theorem enables us to construct E, when A and C are given, by a simple process :

Draw the tangents to the cubic at the points A and C, and let these tangents intersect the curve again in A′ and C′ respectively. The line A′C′ intersects the cubic again at a point E′. The points of contact of the tangents drawn from E′ are the required points E. But, in general, four tangents can be drawn from E′ to the curve, and the point of contact of any one of these tangents will be the required point E. But only three of these points will give the solution of the problem; for AC intersects the cubic at a third point D, the tangent at which also passes through E′. In fact, the line AC taken twice may be regarded as a conic having simple contact with the cubic at the points A, C and D. Consequently, there are three different systems of doubly infinite number of conics which have simple contact with the curve at three different points.

18. If two of the points A, C, E (say A, C) coincide, the conic has a four-pointic contact at A and a simple contact at E. In consequence, the points A′ and C′ coincide, and E′ is the tangential of the point A′, or as we say, E′ is the second tangential of A.

Hence to construct a conic having a four-pointic contact at A and a simple contact elsewhere, we proceed as follows :—

Draw the tangent at A to the cubic, and let A′ be the tangential of A, and E′ the tangential of A′, *i.e.*, the second tangential of A. Then the four points of contact of the tangents drawn from E′ to the curve will be the required points E. But one of these points is A′, which is the tangential of A. Hence, the other three points of contact will give the solution of the problem.

Thus, *three* conics can be drawn having a four-pointic contact at any given point of a cubic and a simple contact elsewhere.

Ex. A conic has a four-pointic contact with a cubic at A and meets the curve again at E and F. EF meets the cubic at the second tangential of A.

19. Pencil of conics through four points on a cubic:

A cubic is described through four fixed points on a cubic. The chord joining the two remaining intersections of the conic with the cubic passes through a fixed point on the curve.

Let A, B, C, D be four given points on a cubic and let the sides AB, BC, CD, DA respectively be denoted by x, y, z and w.

Then the equation of the cubic is $xzu=ywv$, where u and v are linear functions of the variables, and the equation of the conic may be written as $xz=kyw$.

Combining these two equations, we obtain $v=ku$, which is the equation of the line through the two remaining intersections of the cubic and the conic, and for all values of k, this line always passes through the fixed point. $u=v=0$ on the cubic. This point is co-residual of the four given points A, B, C, D.

If A, B, C, D are the points of contact of the tangents drawn from any point P, then from the preceding article it follows that the fixed point O is the tangential of P.* The points P, O, A, B, C, D and the diagonal points of the quadrilateral ABCD form nine associated points.

20. Theorem:

If a conic osculates a cubic at two distinct points, the chord of contact intersects the curve again in a point of inflexion.

Consider a conic which has a contact of the second order with a cubic at the two points A and B. The conic

* Schröter, *Die Theorie der ebenen Kurven 3 Ordnung* (1888), p. 108.

passes through three consecutive points at A and at B Let A′B′ and A″B″ be two other lines consecutive to AB, such that they pass through these consecutive points at A and B. These lines intersect the cubic in three other consecutive points C, C′, C″.

Consider the following three cubics passing through the eight points A, A′, A″; B, B′, B″; C, C′:—

(1) The given cubic,

(2) The cubic consisting of the three lines A′B′, AB, A″B″,

(3) The conic and the line CC′.

Therefore they must pass through a ninth common point C,″ *i.e.*, the line CC′ must pass through C″. Thus three consecutive points C, C′, C″ are collinear, or, in other words, the point C in which the line AB cuts the cubic again is a *point of inflexion* on the cubic.

The theorem of this article may be stated in a different form:—

Every chord drawn through a point of inflexion intersects a cubic in two other points such that a conic can be described having a three-pointic contact with the curve at those points.

Again, since a cubic has nine points of inflexion, there are nine different systems of conics which have a contact of the second order at two points of a cubic curve.

21. Sextactic points on a cubic:

Definition: A *sextactic point* on a curve is a point, the osculating conic at which has six-pointic contact with the curve at that point.

From what has been said in the preceding article, it follows that when the points A and B coincide, the conic has six-pointic contact with the cubic at A, which now becomes the point of contact of a tangent drawn from a point of inflexion.

Hence, the points of contact of tangents drawn from a point of inflexion are sextactic points on the cubic.*

Since there are nine points of inflexion on a non-singular cubic and from each point of inflexion only three tangents can be drawn to the curve, it follows that there are *twenty-seven* sextactic points on a cubic curve.

Hence we may state the following theorem :—

The points of contact of tangents drawn from the points of inflexion are sextactic points, and there are twenty-seven such points on a non-singular cubic curve.

If, however, the cubic has a node, there are only three points of inflexion and from each point only one tangent can be drawn.

Thus, *there are only three sextactic points on a nodal cubic.*

It will be noticed that there can be no such point on a cuspidal cubic.

Halphen considered the case of coincidence of $3m$ points of intersection of an m-*ic* with a cubic, *i.e.*, of $3m$-pointic contact. For $m=1$, we get the nine inflexions, for $m=2$, we get the 27 sextactic points, and when $m=3$, Halphen calls the point "*coincidence point*"—Math. Ann., Vol. 15 (1879).

The above theorem may very easily be proved by applying the principles of residuation.

Let P be a point of inflexion and A the point of contact of a tangent drawn from P.

Then $[3P]=0$ and $[P+2A]=0$, or, $[3P+6A]=0$.

* Cayley—"*On the conic of five-pointic contact at any point of a plane curve*"—Phil. Trans., Vol. 49 (1859), pp. 371-400. The theory of sextactic points on any curve has been discussed by Cayley in his paper—*On the sextactic points of a plane curve*—Phil Trans., Vol. 155 (1865), pp. 545-578.

Hence by the subtraction theorem, $[6A]=0$, or, A is a *sextactic* point.

It will be shown hereafter that the points of contact of tangents drawn from an inflexion are collinear. Therefore the sextactic points lie by three on nine right lines.

Ex. 1. Prove that the points where the harmonic polars intersect the cubic are the sextactic points.

Ex. 2. Shew that of the sextactic points nine are real and 18 imaginary.

22. THEOREM :

If a conic osculates a cubic at two distinct points, one of which is a sextactic point, then the other point must also be a sextactic point.

This is easily proved by residuation as follows :

Let P and Q be the two points of osculation. Then we have

$$[3P+3Q]=0$$

But, P being a sextactic point, $[6P]=0$.

Therefore, by the theorems of multiplication and subtraction, we obtain $[6Q]=0$, which shows that six consecutive points at Q are the complete intersections of the cubic with acurve, which is evidently a conic.

The properties established in this and the preceding articles can easily be obtained by expressing the co-ordinates of a point on the curve in terms of elliptic functions, as will be shown in a subsequent Chapter.

Ex. 1. Through four given points on a cubic, four conics can be drawn touching the curve elsewhere.

Ex. 2. Shew that the tangents at the four points of contact in *Ex.* 1. meet the cubic at the same point.

23. Chasles' method of generation of cubic curves:

The theorem of § 19 suggests a method of generating the cubic curve by points.

The conics through A B C D and the lines through O form two projective pencils, which generate the cubic curve.

Let A, B, C be three fixed points and O any other point on the cubic. Then any line through O meets the cubic in two other points E, F, and the conic ABCEF meets the cubic again in a fixed point D, and the line OEF and the conic ABCDEF have a one-to-one correspondence.

From this theorem of Cremona, Chasles gave a method of constructing a cubic through nine given points.*

Let $L+\lambda M=0$ be a pencil of lines, whose vertex is the point $L=M=0$. A pencil of conics, projectively related with it, is $\phi+\lambda\psi=0$. If we eliminate λ between these two equations, we obtain

$$f \equiv L\psi - M\phi = 0$$

which evidently represents a cubic curve passing through nine points, namely, the four points of intersection of ϕ and ψ, the points where L intersects ϕ, and M intersects ψ, and the point $L=M=0$.

It will be noticed that the point $L=0$, $M=0$ is the coresidual of the system of four points common to ϕ and ψ, and is called the point *opposite* to the four points.

Hence we obtain the following:—

If four points on a cubic be taken as the base of a pencil of conics, the opposite point is the vertex of a pencil of lines, projective with the pencil of conics, the two pencils together generate the curve.

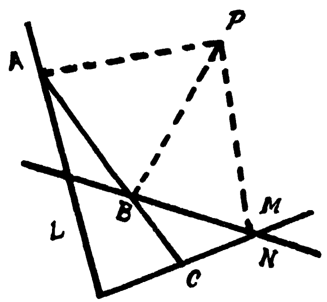
* Chasles—*Construction de la courbe du troisieme ordre determineé par neuf points*—Comp. rend., Vol. 36 (1853), p. 943.

Chasles gave another construction in Comp rendus, Vol. 37 (1853), p. 275.

24. GRASSMANN'S CONSTRUCTION :

*The locus of a point, such that the lines joining it to three fixed points intersect three fixed lines in three collinear points, is a curve of the third order.**

Let $P(x, y, z)$ be the variable point, and $a(a_1, a_2, a_3)$, $b(b_1, b_2, b_3)$ and $c(c_1, c_2, c_3)$ be three fixed points, with reference to the three fixed lines as the sides of the triangle



of reference, then the equation of the line Pa is—

$$X(ya_3-za_2)+Y(za_1-xa_3)+Z(xa_2-ya_1)=0 \qquad \dots \quad (1)$$

This meets the side L ($X=0$),

where
$$\frac{X}{0}=\frac{Y}{xa_2-ya_1}=\frac{-Z}{za_1-xa_3} \qquad \dots \quad (2)$$

Similarly, the other two points of intersection are given

by
$$\frac{X}{xb_2-yb_1}=\frac{Y}{0}=\frac{-Z}{yb_3-zb_2} \qquad \dots \quad (3)$$

and
$$\frac{X}{zc_1-xc_3}=\frac{-Y}{yc_3-zc_2}=\frac{Z}{0} \qquad \dots \quad (4)$$

* Grassmann—Crelle, Bd. 31 (1846), p. 111, Bd. 36 (1848), p. 117, etc. For different methods of construction, see Cayley—Journal de Math. pures et appliques t. 9(1). (1844) p. 287, Schröter—Math. Ann, Bd. 5 (1872), p. 50. Hurwitz—Crelle, Bd. 107 (1891), p 141.

$\therefore$ If these three points are collinear, we must have

$$\begin{vmatrix} 0 & (xa_2-ya_1), & -(za_1-xa_3) \\ xb_2-yb_1, & 0 & -yb_3-zb_2) \\ zc_1-xc_3, & -(yc_3-zc_2) & 0 \end{vmatrix} = 0 \quad \dots \quad (5)$$

From equation (5) it follows that the locus is of order three and passes through the points a, b and c, and through the vertices of the triangle of reference. It can be shown that the curve passes also through the points where the lines bc, ca, ab respectively meet the sides.

Hence the cubic is determined by these nine points. It can further be shown that all curves of the third order may be generated in this manner. Of the nine points, a, b, c and the vertices of the triangle of reference form two triangles, whose vertices are situated on the curve and whose sides respectively meet in three points on the curve. If now we consider these last three points as arbitrarily given on the curve, it can be shown that there are four triangles, of which the vertices a, b, c are situated on the curve of the third order, and whose sides pass separately through three given points on the curve.

For further information on the generation of cubic curves, the student should consult the papers cited before. The converse theorem that every cubic can be constructed by this method has been proved by Clebsch—Math. Ann., Bd. 5 (1872), p. 425.

Ex. 1. Indicate the method how a cubic can be generated by means of nine given points.

Ex. 2. Given nine points, construct by ruler the intersection of the cubic through them with the line through two of the points, or the conic through five.

Ex. 3. Shew that a cubic with a double point can be generated by means of two projective pencils of lines and conics, such that the vertex of the pencil of lines is a base-point of the pencil of conics.

Section II : Harmonic Properties.

25. POLAR CONIC AND POLAR LINE:

Every point has a polar conic and a polar line with respect to a cubic. The polar conic is the first polar which, therefore, passes through the points of contact of the tangents drawn from the point to the curve.

The equation of the polar conic of a point (x', y', z') is

$$x'\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+z'\frac{\partial f}{\partial z}=0,$$

and that of the polar line is—

$$x\frac{\partial f}{\partial x'}+y\frac{\partial f}{\partial y'},+z\frac{\partial f}{\partial z'}=0.$$

The equation of the polar conic in the expanded form may be written as—

$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=0,$$

where, a', b', c'... represent the second differential co-efficients in which (x', y', z') have been substituted for (x, y, z).

The first polar of any point is a conic which meets the cubic in six points, which are the points of contact of the tangents that can be drawn from the point to the cubic. Thus the class of a non-singular cubic is *six*. If, however, the cubic has a node, the class reduces to *four* only, and the class of a cuspidal cubic is only *one*.*

Ex. 1. The tangentials of the six points of contact of the tangents drawn from any point to a cubic lie on a conic having double contact with the first polar of the point.

Ex. 2. Show that the points of contact of the two conics in Ex. 1. are the points where the polar line of the point meets it. [Milinowski—Crelle, Bd. 89 (1880), p. 145].

* Theory of Plane Curves, Vol. I, §§ 120-121.

26. Four Poles of a right line:

If A and B be any two points on the line, their first polars pass through the poles, and since the first polars are conics, they intersect in four points, which are the *poles* of the line AB.

It is evident that the polar conics of points on a right line form a pencil of conics through the four poles of the line.

Conversely, any conic through the four poles of a line is the polar conic of some point on the line.

If, however, the cubic has a node, all the polar conics pass through the node, and therefore the polar conics of A and B intersect in only *three* other points, which are the *poles* of the line.

In the case of a cuspidal cubic, the polar conics pass through the cusp and touch the cuspidal tangent. Therefore they intersect in *two* other points only, which are the *poles* of the line.

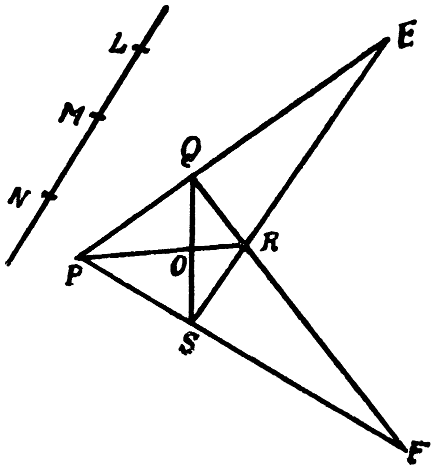
If the cubic reduces to three right lines, all polar conics pass through the vertices of the triangle formed by them, and every right line has only *one* pole.

In this case the point and the line are called respectively the *pole* and *polar* *w.r.t.* the triangle.

27. The three pairs of right lines through the four poles of a line with respect to a non-singular cubic are the polar conics of three points L, M, N on the line. These three points must therefore lie on the Hessian of the cubic. Hence L, M, N are the points in which the line intersects the Hessain, which is therefore a curve of the third order.

Let the four poles of the line AB be P, Q, R and S, and let the pairs of lines PQ, SR; PS, QR, and PR, QS, which constitute the polar conics of the three points L, M, N on AB, intersect in the points E, F, O respectively. Then the polar conics of O, E, F break up into pairs of right lines intersecting at L, M, N respectively.

Suppose the two poles S and R coincide. Then O and F coincide with R and therefore the corresponding points M and N must coincide.



Thus the line LMN becomes a tangent to the Hessian at the point M, corresponding to O or F. The point L corresponds to E. Hence, if a line touches the Hessian, two of the points O, E, F coincide at one of the poles, which corresponds to the point of contact. This is true for every tangent to the Hessian, and we obtain the theorem :

The Hessian is the envelope of a line two of whose poles coincide.

The four poles of a tangent to the Hessian are the points R counted twice and the points P and Q. But P and Q become the points of contact of PF and QF with their envelope, which is the *Cayleyan.**

We may deduce the following as a particular case :— *The polar line of any point on the Hessian with respect to the cubic is the tangent to the Hessian at the corresponding point.* This will be directly proved in a subsequent article.

* Theory of Plane Curves, Vol I, § 100.

28. THEOREM:

The points of contact of the tangents, drawn to a cubic from a point of inflexion, lie on a right line.

The equation of a cubic having the vertex A of the triangle of reference as a point of inflexion is—

$$f \equiv x^2 u_1 + x u_1 v_1 + u_3 = 0.$$

The first polar or the polar conic of A is—

$\frac{\partial f}{\partial x} = u_1(2x + v_1) = 0$, *i.e.*, the polar conic breaks up into two right lines, one of which $u_1 = 0$ is the tangent at A, and the other line $2x + v_1 = 0$ passes through the points of contact of the tangents drawn from A.

DEFINITION: The line $2x + v_1 = 0$, which passes through the points of contact of the tangents drawn from a point of inflexion, is called the *Harmonic Polar* of the point of inflexion.*

29. THEOREM:

The chords of contact of tangents drawn from any point of a cubic are harmonic conjugates of the tangent at their intersection and the line joining the intersection with the point.

Let L, M, L′, M′ be the four tangents drawn from A, whose chords of contact are respectively P, Q, intersecting at the diagonal point c_3 (§, 13), where N is the tangent. Then the equation of the cubic can be written in either of the forms—

$$LMN - P^2R = 0, \quad L'M'N - Q^2R = 0,$$

where R is the tangent at A.

* This line is called the *harmonic polar* simply to distinguish it from the ordinary polar line of the point A, which, in the present case, is the inflexional tangent at A. Jonquières—*Mel de geométrie pure*. p. 228.

We have then the identity—

$$N(LM-L'M') \equiv (P^2-Q^2)\, R,$$

the right-hand side of which represents three right lines.

Consequently the left-hand side will also represent three right lines. Evidently, N is one of $P \pm Q$, and R is a factor of $(LM-L'M')$. Hence the other factor of $LM-L'M'$ must be one of $P \pm Q$., *i.e.*, the line Ac_3. But the four lines P, Q, $P \pm Q$ form a harmonic pencil, which proves the proposition.

30. MACLAURIN'S THEOREM:

Any line drawn through a point of a cubic is cut harmonically by the curve and the chords of contact of tangents drawn from the point

Let the line through A meet the cubic in two other points P and Q and the chords of contact of tangents drawn from A at the points L and M respectively. Then, if the chords intersect at the point O and the line APQ meets the tangent at O in R, then ALRM is a harmonic range, so that

$$\frac{1}{LA}+\frac{1}{LR}=\frac{2}{LM}$$

By another theorem of Maclaurin,* since the line APLMQ through L meets the cubic in A,P,Q and the tangents at three points (collinear with L) in A, A, R, we have—

$$\frac{1}{LP}+\frac{1}{LA}+\frac{1}{LQ}=\frac{1}{LA}+\frac{1}{LA}+\frac{1}{LR}$$

$$i.e.\ \frac{1}{LP}+\frac{1}{LQ}=\frac{1}{LA}+\frac{1}{LR}=\frac{2}{LM}$$

i.e., PQLM is a harmonic range.

The theorem still holds, if the cubic has a double point, and consequently, when only two tangents can be drawn. In this case the line joining the double point to the point where the chord of contact meets the curve again is to be taken as the second chord.

* Theory of Plane Curves, Vol. I. § 70.

31. THEOREM:

Every chord drawn through a point on a cubic is cut harmonically by the curve and the polar conic of the point.

Let $f \equiv bx+cy+dx^2+exy+fy^2+gx^3+\ldots \quad =0$

be the equation of the curve referred to the point O as origin.

Then the polar conic of O is given by—

$$\frac{\partial f}{\partial z} \equiv 2(bx+cy)+dx^2+exy+fy^2=0$$

Let the line $\frac{x}{l}=\frac{y}{m}=r$ intersect the cubic in P and Q and the polar conic in R.

Then, from the equation of the cubic, we obtain—

$$\frac{OP+OQ}{OP \cdot OQ}=\frac{1}{OP}+\frac{1}{OQ}=-\frac{dl^2+elm+fm^2}{bl+cm}$$

and from the equation of the polar conic, we have—

$$OR=-\frac{2(bl+cm)}{dl^2+elm+fm^2}$$

whence, $\frac{1}{OP}+\frac{1}{OQ}=\frac{2}{OR}$, *i.e.*, the points O, P, R, Q form a harmonic range.

If, however, the point O is a point of inflexion on the curve, the polar conic consists of the inflexional tangent and the harmonic polar, and we obtain the theorem.—

Every chord drawn through a point of inflexion on a curve is divided harmonically by the curve and the harmonic polar; or in other words,

*If radii vectores be drawn through a point of inflexion, the locus of the harmonic means will be a right line (the harmonic polar).**

* This theorem is due to Maclaurin—*Die Linearum Geometricarum Proprietalibus Generalibus.*, Sec. III, Prop. 9.

Ex. 1. Find the harmonic polar of $(-1, 1, 0)$ for the cubic

$$(x+y+z)^3+6mxyz=0$$

Ex. 2. Two chords OPQR and OP′Q′R′ meet the cubic in P,Q; P′,Q′ and the polar conic of O in R, R′. Shew that the lines PP′, QQ′, RR′ meet at the same point. Shew also that the tangents at P,Q,R, are concurrent.

Ex. 3 Shew that the harmonic polar of the point $(0,-1, 1)$ for the cubics of the system $x^3+y^3+z^3+6mxyz=0$ is the same.

32. THEOREM:

The node of a nodal cubic is the pole of the line joining its three points of inflexion.

Taking the inflexional tangents as the sides of the triangle of reference, the equation of the cubic can be written as

$$xyz=(lx+my+nz)^3\equiv u^3.$$

If this has a double point (x', y', z'), the first differential co-efficients must vanish at that point, *i.e.*, we must have

$$y'z'=3lu'^2, \quad z'x'=3mu'^2, \quad x'y'=3nu'^2,$$

where $$u'=lx'+my'+nz'.$$

$$\therefore \qquad lx'=my'=nz' \qquad \dots \quad (1)$$

The polar line of any point (x', y', z') is—

$$x(y'z'-3lu'^2)+y(z'x'-3mu'^2)+z(x'y'-3nu'^2)=0 \qquad \dots \quad (2)$$

If this is to be the same line as $lx+my+nz=0$, we must have

$$\frac{y'z'-3lu'^2}{l}=\frac{z'x'-3mu'^2}{m}=\frac{x'y'-3nu'^2}{n}$$

or, $$\frac{y'z'}{l}=\frac{z'x'}{m}=\frac{x'y'}{n}, \quad i.e., \quad lx'=my'=nz'$$

Therefore, from equations (1), the pole (x', y', z') of $lx+my+nz=0$ is the double point, which proves the proposition.

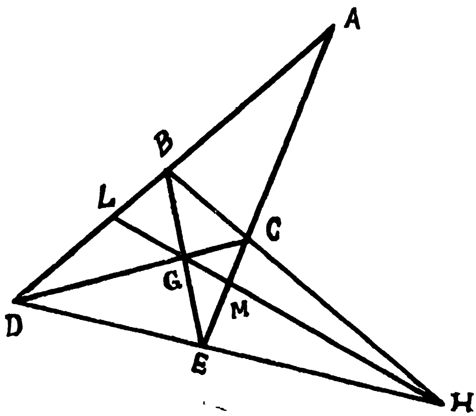
33. Properties of the Harmonic Polars :

A point of inflexion on a cubic possesses, with regard to its harmonic polar, properties analogous to those of poles and polars in conic sections.

Theorem : *If two chords be drawn through a point of inflexion to meet a cubic in four points and the extremities be joined directly and transversely, the two points of intersection lie on the harmonic polar.*

This follows immediately from the harmonic properties of a quadrilateral. For, let A be a point of inflexion, and let AB and AC meet the cubic in B, D and C, E respectively. Let BE and CD meet in G and BC, DE intersect at H. Then GH is the harmonic polar of A.

For, if GH meets AB and AC in L and M respectively, (A B L D) and (A C M E) are harmonic. Hence, GH is the locus of harmonic means between AB and AD.



We may easily deduce the following :—If a line intersects a cubic in three points A, B, C, the lines joining A, B, C to a point of inflexion O meet the cubic again in three collinear points A′, B′, C′, and the two

lines ABC, A'B'C' meet the harmonic polar in the same point.

If, however, A, B, C coincide, then A', B', C' also coincide, and it follows that the three points of inflexion O, A, A' are collinear, and the tangents at A and A' meet on the harmonic polar of O, *i.e.*,

The inflexional tangents at two of three collinear points of inflexion meet on the harmonic polar of the third.

This theorem may be analytically proved as follows :—

Taking the equation of the cubic in the form

$$xyz=(lx+my+nz)^3,$$

we see that x, y, z are the inflexional tangents, and the point $(0, -n, m)$ is an inflexion on $x=0$, whose harmonic polar is the line $my-nz=0$, and this evidently passes through the intersection of the other inflexional tangents y and z.

As a particular case of the above theorem, we may deduce the following :—*Tangents at the extremities of any chord drawn through a point of inflexion intersect on the harmonic polar.*

This follows by considering that the lines AB and AC coincide, when BC and DE become tangents at B and D.

Again, if we consider that B and D coincide, *i.e.*, AB becomes a tangent, then the harmonic polar passes through the point of contact of AB. For, (ABLD) is harmonic, and when B coincides with D, it must coincide with L.

Hence we obtain the theorem that *the harmonic polar paeses through the points of contact of the tangents drawn from the point of inflexion.*

Now, since the harmonic polar intersects the cubic in three points, three tangents can be drawn from a point of inflexion, and their points of contact lie on a right line.

34. THEOREM :

The harmonic polars of three points of inflexion which lie on a right line pass through the pole of the line with regard to the traingle formed by the inflexional tangents.

Taking the equation of the cubic in the form

$$xyz=(lx+my+nz)^3\equiv u^3$$

the harmonic polars of the three points of inflexion are respectively

$$my-nz=0,\quad nz-lx=0,\quad lx-my=0,$$

which evidently meet at the point $lx=my=nz$, *i.e.*, at the point $(1/l, 1/m, 1/n)$ which is the pole of the line

$$lx+my+nz=0$$

with regard to the triangle of reference.*

When the curve is a nodal cubic, the three harmonic polars pass through the node.

For the harmonic polars meet at the point $lx=my=nz$ which is the node (§ 32).

Since only one tangent can be drawn from a point of inflexion to a nodal cubic, the harmonic polar is the line joining the node to the point of contact of the tangent. When the curve has a cusp, the harmonic polar is the cuspidal tangent.

35. INFLEXIONAL TRIANGLES :

A non-singular cubic has nine points of inflexion, which have definite positions relative to one another, namely, the line joining any two passes through a third. Hence, it follows that through each point there pass four lines, each of which passes through two of the remaining eight points of inflexion. These lines are called *inflexional lines*.

* Scott—*loc. cit.*, § 23.

In actual arrangement, each line is counted thrice, and consequently, there exist only $\frac{1}{3}.9.4=12$ such lines. Denoting the points by the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and the inflexional line through three points i, j, k by the symbol $L_{i,j,k}$, for example, L_{123} denoting the line through 1, 2, 3, we see that the three lines L_{123}, L_{456}, L_{789} are such that they contain all the nine points of inflexion. They form a triangle called an *inflexional triangle.*

There are *four* such triangles, whose sides pass through all the nine points of inflexion. Thus we obtain the following theorem* :—

The nine points of inflexion lie, three by three, on twelve lines and these lines can be grouped, three by three, to form four inflexional triangles such that each group contains all the nine points of inflexion.

Ex. 1. Show that there is only one real inflexional triangle.

Ex. 2. Shew that a pencil of cubics pass through the nine inflexions of a cubic.

36. Configuration of the harmonic polars† :

The nine harmonic polars are the same for all curves of the pencil passing through the nine points of inflexion. We shall therefore obtain polars of the points of inflexion with respect to all these curves, if we consider an inflexional triangle as the primitive curve and construct the polar conic of a point of inflexion with respect to this triangle. The polar conic in question consists of the side of the triangle which passes through the point, and of the polar line of the point of inflexion with respect to two other sides of the triangle. If, therefore, we join the point to the vertex of one of the four inflexional triangles, opposite to the side on which it lies, and find the fourth harmonic to this line

* Clebsch—Crelle, Bd. 63 (1864), p. 120.

† Hesse—Crelle, Bd. 28 (1844) p. 97 and Bd. 38 (1849) p. 257.

and the two sides of the triangle, this last line is the harmonic polar.

It follows, therefore, that the harmonic polars of three collinear points of inflexion are concurrent in the vertex opposite to the inflexional line. Thus, through a vertex of each inflexional triangle, there pass three harmonic polars, and consequently each harmonic polar passes through one vertex of each of the four inflexional triangles. We thus obtain the theorem:

The twelve vertices of the inflexional triangles are situated, four by four, on the nine harmonic polars. The harmonic polars of three points of indexion intersect at a vertex of the triangle corresponding to the inflexional line; to the sides of a triangle there correspond the opposite vertices.

Thus we see that the nine harmonic polars form a dual system to the nine points of inflexion. They determine four triangles, through the vertices of each passing all the nine harmonic polars. These triangles are identical with the infiexional triangles, and each harmonic polar passes through one vertex of each of these triangles. In fact, to the pencil of order-cubics through the points of inflexion there corresponds, by the principle of duality, a system of class-cubics * with these nine harmonic polars as common cuspidal tangents. This system of class-cubics, as we shall see later, is the system of Cayleyan curves of the cubics of the pencil.

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* A class-cubic is a curve of the third class, but not necessarily of the third order. *Vide* C. A. Scott, Modern Ana. Geo., § 67.

CHAPTER II

Canonical Forms

37. Reduction of the general equation:

The general equation of a ternary cubic contains ten arbitrary constants, six of which can be made to vanish by a proper choice of the triangle of reference. Thus the equation of a certain class of cubic curves can be put into the form

$$ax^3+by^3+cz^3+lxyz=0.$$

Then, by a linear transformation of the type

$$X=a^{\frac{1}{3}}x, \qquad Y=b^{\frac{1}{3}}y, \qquad Z=c^{\frac{1}{3}}z,$$

the equation reduces to the form—

$$X^3+Y^3+Z^3+kXYZ=0.$$

This is called a *canonical form* of the equation and is due to Hesse.

This transformation is possible only in the case of a cubic whose discriminant does not vanish, *i.e.*, in the case of a non-singular cubic. It is, however, proved in works on Algebra* that the equation of such a cubic can be reduced to the form

$$x^3+y^3+z^3+6mxyz=0 \qquad \dots \quad (1)$$

where x, y, z are linear functions of the variables and may be regarded as the co-ordinates of a point referred to a new triangle whose sides, referred to the original triangle, are

$$x=0, \qquad y=0, \qquad z=0.$$

The number of independent constants involved in this equation is $3\times 3+1=10$, and therefore, is the same as that of the co-efficients in the general equation. Thus we have an *a priori* indication of the possibility of such a transformation.

* E. B. Elliot, Algebra of Quantics, § 29.

38. GEOMETRICAL RELATION OF THE CURVE WITH THE NEW TRIANGLE OF REFERENCE:

The equation $x^3+y^3+z^3+6mxyz=0$ can be written as

$$x^3+y^3-8m^3z^3-3xy(-2mz)=-(1+8m^3)z^3;$$

or, $$(x+y-2mz)\ (\omega x+\omega^2y-2mz)\ (\omega^2x+\omega y-2mz)$$
$$=-(1+8m^3)z^3,$$

where ω is an imaginary cube root of unity.

Now, if $1+8m^3\neq0$, the equation can be put into the symbolic form $P.Q.R=z^3$, where P, Q, R are linear functions of the variables and are evidently the inflexional tangents at the three points where $z=0$ cuts the curve. Hence $z=0$ meets the curve in three of its points of inflexion. Similarly, since the equation is symmetrical in x, y and z, the sides $x=0$ and $y=0$ also meet the curve each in three points of inflexion.

Thus the sides of the new triangle of reference are the three lines on which the nine points of inflexion lie, three by three.

Hence, in order to reduce the equation of a non-singular cubic to its canonical form, an inflexional triangle is taken as the triangle of reference.

The inflexions on $x=0$ are determined by the equations

$$x=0 \quad\text{and}\quad y^3+z^3=0$$

i.e., $$(y+z)\ (y+\omega z)\ (y+\omega^2z)=0$$

Since these co-ordinates are independent of m, it follows that all cubics included in the canonical form for different values of m meet the three sides of the triangle of reference in the same nine points, which are inflexions on all, provided

$$1+8m^3\neq0.$$

When $1+8m^3=0$, $m=-\frac{1}{2}$ and the cubic reduces to three right lines. Again, if $m=1$, it is seen that the

three real inflexional tangents are concurrent. Hence the equation represents a non-singular cubic, except when $m=1$ or $-\frac{1}{2}$.

Ex. 1. The polar line of a point P *w.r.t.* the polar conic of Q is the same as the polar line of Q *w.r.t.* the polar conic of P.

Ex. 2. Obtain the equation of a non-singular cubic, not having the real inflexional tangents concurrent, in the form

$$x^3+y^3+z^3=k\,(x+y+z)^3$$

Ex. 3. Shew that there are *twelve* points in the plane such that the polar line of each is the same *w.r.t.* all cubics of the system. [These points are called the *critic centres* of the system, [*cf.* Vol. I, Ch. XIV, § 292].

39. Co-ordinates of the nine inflexions:

The equation of the cubic can be written in the three following forms:—

$$(-2mx+y+z)(-2mx+\omega^2y+\omega z)(-2mx+\omega y+\omega^2 z)$$
$$=-(1+8m^3)x^3$$

$$(x-2my+z)(\omega x-2my+\omega^2 z)(\omega^2 x-2my+\omega z)=-(1+8m^3)y^3$$

$$(x+y-2mz)(\omega x+\omega^2 y-2mz)(\omega^2 x+\omega y-2mz)=-(1+8m^3)z^3$$

The co-ordinates of the nine points of inflexion are therefore obtained as follow:—

$$\begin{array}{ccc}
(0.\ 1,\ -1) & (-1,\ 0,\ 1) & (1,\ -1,\ 0) \\
(0,\ \omega,\ -1) & (-1,\ 0,\ \omega) & (\omega,\ -1,\ 0) \\
(0,\ \omega^2, -1) & (-1,\ 0.\ \omega^2) & (\omega^2, -1,\ 0)
\end{array}$$

The nine linear factors on the left-hand sides give the inflexional tangents.

It appears, therefore, that of these nine points of inflexion only *three* are real, each lying on a side of the triangle of reference.

40. CONFIGURATION OF THE INFLEXIONS:

The equation of the curve can also be written in the form—

$$f \equiv x^3+y^3+z^3-3xyz+3(2m+1)xyz=0 \quad \dots \quad \dots \quad (1)$$

and consequently, in the three following forms:—

$$\left.\begin{array}{l}(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)+3(2m+1)xyz=0 \\ (\omega x+y+z)(x+y+\omega z)(\omega^2 x+y+\omega^2 z)+3(2m+\omega)xyz=0 \\ (\omega^2 x+y+z)(x+\omega^2 y+z)(\omega x+\omega y+z)+3(2m+\omega^2)xyz=0\end{array}\right\} (2)$$

or in the abridged forms:

$$\left.\begin{array}{l}a_1 a_2 a_3+\lambda xyz=0 \\ b_1 b_2 b_3+\mu xyz=0 \\ c_1 c_2 c_3+\nu xyz=0\end{array}\right\} \quad \dots \quad \dots \quad (3)$$

Hence we see that the inflexions lie on the twelve lines

(*i*) $x, y, z,$ (*ii*) $a_1, a_2, a_3,$

(*iii*) $b_1, b_2, b_3,$ (*iv*) $c_1, c_2, c_3.$

The four groups of three lines each form four triangles which are called *inflexional triangles.* We may express the grouping of these nine points by the following scheme:—

$$\begin{array}{ccccc} & b_1 & b_2 & b_3 & \\ c_1 & 1 & 2 & 3 & x \\ c_2 & 4 & 5 & 6 & y \\ c_3 & 7 & 8 & 9 & z \\ & a_1 & a_2 & a_3 & \end{array} \quad \dots \quad (4)$$

where the nine inflexions are designated by the numbers 1, 2, 3, ... 9.

The four inflexional triangles are each formed by the three lines which pass through

(I) the three points in each horizontal line,

(II) the three points in each vertical line,

(III) the three points which give a positive term in (4 considered as a determinant,

(IV) the three points which give a negative term in the same determinant.

For instance, on b_1 lie the three points 1, 5, 9; on b_2 lie 2, 6, 7; on b_3 lie 3, 4, 8, and so on.

Ex. 1. The twelve inflexional lines intersect in the nine inflexions and the twelve critic centres of the system of cubics.

Ex. 2. Find the co-ordinates of these critic centres.

Ex. 3. Shew that a quartic through the critic centres of a cubic has an inflexion at each of the critic centres.

Ex. 4. Shew that the lines joining an inflexion with the remaining eight inflexions form an equianharmonic pencil.

Ex. 5. The intersection of an inflexional tangent with the corresponding harmonic polar lies on the Hessian.

41. Transformation of canonical forms :

$$f \equiv x^3 + y^2 + z^3 + 6mxyz = 0$$

is the equation of the cubic, referred to the triangle whose sides are

$$x=0, \quad y=0, \quad z=0.$$

To obtain its equation with reference to another inflexional triangle, we put

$$X \equiv x+y+z \qquad Y \equiv x+\omega y+\omega^2 z$$

$$Z \equiv x+\omega^2 y+\omega z$$

Let $\qquad \phi \equiv x^3+y^3+z^3, \qquad$ and, $\qquad \psi \equiv xyz.$

Then, $\phi' \equiv X^3+Y^3+Z^3 = 3(x^3+y^3+z^3)+18xyz = 3\phi+18\psi.$

$$\psi' \equiv XYZ = (x^3+y^3+z^3)+3(\omega+\omega^2)xyz = \phi-3\psi.$$

whence $\quad 9\phi = (\phi'+6\psi') \quad$ and $\quad 27\psi = \phi'-3\psi')$.

But the equation of the curve is—

$$\phi+6m\psi=0, \quad \text{or,} \quad 9\phi+54m\psi=0$$

which gives $\quad \phi'+6\psi'+2m(\phi'-3\psi')=0$

or, $\quad (1+2m)\phi'+6(1-m)\psi'=0$

i.e., $\quad (1+2m)(X^3+Y^3+Z^3)+6(1-m)XYZ=0.$

Ex. The four inflexional lines through an inflexion are equianharmonic.

42. Equations of the harmonic polars :

The polar conic of the inflexion $(0, 1, -1)$ on the side $x=0$ is

$$(y^2+2mxz)-(z^2+2mxy)=0$$

i.e., $\quad (y-z)(y+z-2mx)=0.$

Therefore, $y-z=0$ is the harmonic polar and $y+z-2mx=0$ is the corresponding inflexional tangent. Similarly, the harmonic polars of the other two points of inflexion on the side

$$x=0 \quad \text{are} \quad y-\omega z=0 \quad \text{and} \quad y-\omega^2 z=0.$$

Thus the nine harmonic polars are—

$$\begin{array}{lll} y-z=0 & z-x=0 & x-y=0 \\ y-\omega z=0 & z-\omega x=0 & x-\omega y=0 \\ y-\omega^2 z=0 & z-\omega^2 x=0 & x-\omega^2 y=0 \end{array}$$

whence it follows at once that the harmonic polars of three collinear points of inflexion meet in a point.

43. HESSE'S THEOREM* :

All cubics described through the nine points of inflexion on a non-singular cubic will have these points for points of inflexion.

The equation † of the Hessian of the cubic

$$F \equiv x^3+y^3+z^3+6mxyz=0 \quad \dots \quad \dots \quad (1)$$

is at once found to be

$$H \equiv m^2(x^3+y^3+z^3)-(1+2m^3)xyz=0 \quad \dots \quad (2)$$

Since the points of inflexion on a curve are its intersections with the Hessian, the equation of a curve passing through the points of inflexion is of the form—

$$F+\lambda H=0$$

where λ is a parameter, *i.e.*, the equation of the system of cubics passing through the points of inflexion is—

$$F+\lambda H \equiv (1+\lambda m^2)(x^3+y^3+z^3)+(6m-\lambda-2\lambda m^3)xyz=0$$

which is of the form

$$x^3+y^3+z^3+6kxyz=0$$

where $$6k \equiv \frac{6m-\lambda-2\lambda m^3}{1+\lambda m^2}.$$

Therefore, the pencil of cubics, to which also belongs the Hessian, has the same inflexions. The system of cubics (1) for different values of m is called a family of *syzygetic* cubics.

* Hesse, Crelle, Bd. 28 (1844), pp. 89-107.

† The form of this equation shows that it belongs to the same system of cubics as are given by (1), where m is considered as a parameter. Since the equation of the Hessian involves m in the third degree, there are three different curves of the system (1) for which $H=0$ is the Hessian.

Ex. 1. A cubic through eight inflexions of a cubic passes also through the ninth.

Ex. 2. The intersection of an inflexional line with the corresponding harmonic polar forms an involution with those inflexions.

Ex. 3. The four lines joining an inflexion with the remaining eight inflexions form an equianharmonic pencil.

44. ANOTHER CANONICAL FORM :

It has been proved that a non-singular cubic has three collinear real Inflexions and three real inflexional tangents. Choosing the inflexional tangents as the sides of the triangle of reference and the line $x+y+z=0$ as the inflexional line, the equation of a cubic can be put into the form

$$(x+y+z)^3+6kxyz=0 \qquad \dots \quad (1)$$

It is to be noticed that this can be effected by a real choice of co-ordinates and that the inflexional tangents are not concurrent.

Ex. 1. Shew that the equation of an acnodal cubic takes the form

$$(x+y+z)^3=27xyz.$$

Ex. 2. Shew that the equation of an acnodal cubic can be reduced to the form

$$x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0.$$

Ex. 3. Find the cubic whose inflexional triangle is self-conjugate *w.r.t.* to the polar conic of the point (1,1,1).

45. A THIRD CANONICAL FORM OF A NON-SINGULAR CUBIC :

Another important canonical form of a non-singular cubic, using the notation of the theory of elliptic functions, is obtained in the form

$$y^2z=4x^3-g_2xz^2-g_3z^3$$

where g_2, g_3 are any constants.

Since a non-singular cubic has always a real point of inflexion, we take the inflexion at B with the inflexional tangent as the side $z=0$ of the triangle of reference, and the harmonic polar as the side $y=0$, while the polar line of the intersection of the inflexional tangent and the harmonic polar as the side $x=0$. Thus the equation of a non-singular cubic is obtained in the form

$$ax^3+by^2z+cxz^2+dz^3=0.$$

Now, making the substitution—

$$(-4b/a)^{\frac{1}{3}}x \text{ for } x \quad \text{and} \quad \frac{c}{b}=g_2, \quad \frac{d}{b}=g_3$$

we obtain $$y^2z=4x^3-g_2xz^2-g_3z^3$$

The harmonic polar $y=0$ meets the cubic in points given by

$$4x^3-g_2xz^2-g_3z^3=0$$

which are all real or one real, according as the discriminant $g_2^3-27g_3^2$ is negative or positive.

If the cubic has a double point, we must have

$$g_2^3-27g_3^2=0$$

and the co-ordinates of the double point are

$$x : y : z=-g_3^{\frac{1}{3}} : 0 : 2.$$

It is easy to see that the double point is a node, cusp or a conjugate point, according as g_3 is negative, zero or positive.

46. THE CANONICAL FORM OF A NODAL CUBIC:

Since a nodal cubic has only three collinear points of inflexion, choosing the nodal tangents and the inflexional line as the sides of the triangle of reference, the equation of a crunodal cubic can be put into the form

$$x^3+y^3+6mxyz=0 \qquad \dots \quad (1)$$

For, the equation of a cubic having a node at C with $x=0$, $y=0$ as nodal tangents can be written as—

$$u_3+kxyz\equiv ax^3+3bx^2y+3cxy^2+dy^3+kxyz=0 \quad \dots \quad (2)$$

Since $z=0$ is to pass through the inflexions, it must meet the cubic (2) and its Hessian

$$kxyz-3(ax^3+bx^2y-cxy^2+dy^3)=0 \quad \dots \quad (3)$$

in the same points, *i.e.*, the equations obtained by putting $z=0$ in (2) and (3), namely

$$ax^3+3bx^2y+3cxy^2+dy^3=0$$

and
$$ax^3-bx^2y-cxy^2+dy^3=0$$

must be the same, which requires that either $a=d=0$, or $b=c=0$.

Excluding the case $a=d=0$, when the cubic reduces to three right lines, we have $b=c=0$, and the cubic becomes

$$ax^3+dy^3+kxyz=0$$

Now put

$$X=a^{\frac{1}{3}}x, \qquad Y=d^{\frac{1}{3}}y \qquad \text{and} \qquad Z=z,$$

and the equation takes the form—

$$X^3+Y^3+6mXYZ=0, \qquad \text{where } 6a^{\frac{1}{3}}d^{\frac{1}{3}}m=k.$$

47. OTHER FORMS OF A NODAL CUBIC:

The equation of a cubic having a node at C can be written as

$$u_2z+u_3\equiv(y+mx)(y+m'x)z$$
$$+(ax^3+3bx^2y+3cxy^2+dy^3)=0$$

where $y+mx$ and $y+m'x$ are the nodal tangents. Next suppose that the vertex B is a point of inflexion, with $z=0$ as the inflexional tangent. Then the co-efficient of

y^3 vanishes, and those of y^2 and y should have the common factor z. This requires that $d=0$, and $c=b=0$.

Thus, the equation reduces to the form—

$$(y+mx)(y+m'x)z+ax^3=0$$

If now the nodal tangents are taken as the harmonic conjugates of $x=0$ and $y=0$, $m=-m'$, and the equation becomes

$$(y+mx)(y-mx)z+ax^3=0$$

which may be written in the form—

$$(y^2-kx^2)z-x^3=0.$$

The equation will represent a cubic, having a node, a cusp, or a conjugate point at C, according as $k \gtreqless 0$.*

Ex. 1. Shew that by a proper choice of homogeneous co-ordinates the equation of a nodal cubic can be reduced to the form

$$z(x^2-y^2)=y(3x^2+y^2).$$

Ex. 2. Show how you can transform the equation of a nodal cubic into the form

$$x^3+y^3=3xyz.$$

Ex. 3. Shew that a nodal cubic and its Hessian are in plane perspective, the node being the vertex and the inflexional line as the axis of perspective.

Ex. 4. A chord of $x^3+y^3=3axy$ subtends a right angle at the node. Shew that the chord meets the cubic again at a fixed point.

Ex. 5. Shew that the tangents at two inflexions of a nodal cubic meet on the harmonic polar of the third inflexion.

* See H. Durège—" Ueber fortgesetztes Tantentenziehen an Kurven 3 Ordnung mit einem Doppel or Rückkehrpunkte."—Math. Ann. Bd. 1, (1869), pp. 509-532.

48. THE CANONICAL FORM OF A CUSPIDAL CUBIC :

In the case of a cuspidal cubic, there is only one point of inflexion. Let the vertex A be the inflexion with $y=0$ as the inflexional tangent. Then the equation of the cubic can be written as—

$$ax^2y+xyv_1+u_3=0 \qquad \dots \quad (1)$$

If the vertex B is a cusp, the equation should not contain y^3 and y^2 ; and if $x=0$ is the cuspidal tangent, the co-efficient of y should be x^2, and the equation reduces to the form

$$ax^2y+bz^3=0.$$

Now, put $X=x$, $Y=ay$ and $Z=-b^{\frac{1}{3}}z$ and the equation becomes

$$X^2Y=Z^3 \qquad \dots \quad (2)$$

Thus we see that every cuspidal cubic can be reduced to the form $x^2y=z^3$, the cuspidal tangent and the inflexional tangent being taken as two sides and the line joining the cusp to the point of inflexion as the third side of the triangle of reference.

It is to be noticed that the Hessian of the euspidal cubic

$$x^2y=z^3$$

is composed of three right lines, of which two coincide with the cuspidal tangent.

Ex. 1. Show that a cuspidal cubic can be projected into a semi-cubical parabola.

Ex. 2. A conic osculates a cubic at two points. Shew that the chord of contact passes through the inflexion.

Ex. 3. The tangents from any point P to a cuspidal cubic meet the curve again in three points. Shew that the tangents at these points are concurrent.

Ex. 4. If the point P describes a line, the point of concurrence moves along a line.

49. COLLINEAR POINTS ON A CUBIC: *

The points where the line

$$\lambda x+\mu y+\nu z=0$$

intersects the cubic

$$x^3+y^3+z^3+6mxyz=0$$

are determined by eliminating z between their equations, *i.e.*, by the equation

$$\nu^3(x^3+y^3)+(-\lambda x-\mu y)^3+6mxy(-\lambda x-\mu y)\nu^2=0;$$

i.e., $$(\nu^3-\lambda^3)\frac{x^3}{y^3}-3(\lambda^2\mu+2m\lambda\nu^2)\frac{x^2}{y^2}-3(\lambda\mu^2+2m\mu\nu^2)\frac{x}{y}$$
$$+(\nu^3-\mu^3)=0. \quad \ldots \quad \ldots \quad (1)$$

If (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the co-ordinates of the three points of intersection, we must have

$$x_1x_2x_3 : y_1y_2y_3=\mu^3-\nu^3 : \nu^3-\lambda^3 \quad \ldots \quad (2)$$

Similarly, by eliminating x between the same equations we obtain

$$z_1z_2z_3 : y_1y_2y_3=\lambda^3-\mu^3 : \nu^3-\lambda^3 \quad \ldots \quad (3)$$

whence

$$x_1x_2x_3 : y_1y_2y_3 : z_1z_2z_3=\mu^3-\nu^3 : \nu^3-\lambda^3 : \lambda^3-\mu^3$$

$$\therefore \; x_1x_2x_3+y_1y_2y_3+z_1z_2z_3=0 \quad \ldots \quad \ldots \quad (4)$$

is a necessary condition that the three points

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2) \text{ and } (x_3, y_3, z_3)$$

on a cubic should lie on a right line.

Again, since the three points are collinear, we may take

$$x_3=\lambda x_1+\mu x_2, \quad y_3=\lambda y_1+\mu y_2, \quad z_3=\lambda z_1+\mu z_2$$

* These theorems on the intersections of a cubic and a line are taken from Cayley—"A memoir on curves of the third order"—Phil. Trans. of the R. Soc. of London, Vol. CXLVII (1857). Coll. Papers, Vol. II No. 146, §§ 29-31.

and substituting these values in the equation of the cubic, we obtain

$$\Sigma(\lambda x_1+\mu x_2)^3+6m(\lambda x_1+\mu x_2)(\lambda y_1+\mu y_2)(\lambda z_1+\mu z_2)=0$$

or, $3\lambda\mu\Sigma x_1 x_2(\lambda x_1+\mu x_2)+3m\lambda\mu\Sigma(\lambda x_1+\mu x_2)(y_1 z_2+y_2 z_1)=0$

whence $\quad \Sigma x_1 x_2 x_3+m\Sigma x_3(y_1 z_2+y_2 z_1)=0$

and consequently, by the preceding condition, we obtain

$$x_3(y_1 z_2+y_2 z_1)+y_3(z_1 x_2+z_2 x_1)+z_3(x_1 y_2+x_2 y_1)=0 \; \ldots \quad (5)$$

Thus the necessary conditions for collinearity of three points on a cubic are given by (4) and (5).

These conditions can also be shown to be sufficient: *i.e.*, the three points satisfying the conditions (4) and (5) are collinear.

For, if not, let the line joining the two points

$$(x_1, y_1, z_1), \qquad (x_2, y_2, z_2)$$

meet the cubic in the point $\quad (x_1', y_3', z_3')$

Then, $\quad \Sigma x_1 x_2 x_3=0 \quad$ and $\quad \Sigma x_1 x_2 x_3'=0$

which shows that (x_3, y_3, z_3) and (x_3', y_3', z_3') are points on the line

$$x_1 x_2 x+y_1 y_2 y+z_1 z_2 z=0 \; \cdots \quad \ldots \quad (6)$$

Similarly, from (5) the two points are found to lie on the line

$$(y_1 z_2+y_2 z_1)x+(z_1 x_2+z_2 x_1)y+(x_1 y_2+x_2 y_1)z=0 \; \ldots \quad (7)$$

whence the two points $\quad (x_3, y_3, z_3) \quad$ and $\quad (x_3'; y_3', z_3')$

coincide at the intersection of (6) and (7), which proves the proposition.

Thus the necessary and sufficient conditions that the three points on the cubic are collinear are—

$$x_1 x_2 x_3+y_1 y_2 y_3+z_1 z_2 z_3=0$$

$$(y_1 z_2+y_2 z_1)x_3+(z_1 x_2+z_2 x_1)y_3+(x_1 y_2+x_2 y_1)z_3=0$$

50. THE CO-ORDINATES OF THE TANGENTIAL POINT:

Let any two, namely A and B, of the three points

$A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ coincide.

Then we have

$$x_1{}^2x^3+y_1{}^2y_3+z_1{}^2z_3=0 \qquad \dots \quad (1)$$

Now, if the equation of the tangent to the cubic at A is

$$\lambda x+\mu y+\nu z=0$$

then $x_1{}^2x_3 : y_1{}^2y_3 : z_1{}^2z_3=\mu^3-\nu^3 : \nu^3-\lambda^3 : \lambda^3-\mu^3$;

Also $\lambda : \mu : \nu=x_1{}^2+2my_1z_1 : y_1{}^2+2mz_1x_1 : z_1{}^2+2mx_1y_1$

Thus we have—

$$\mu^3-\nu^3=k(y_1{}^3-z_1{}^3)\,(y_1{}^3+z_1{}^3+6mx_1y_1z_1-8m^3x_1{}^3)$$

$$=-k(1+8m^3)\,(y_1{}^3-z_1{}^3)x_1{}^3$$

since (x_1, y_1, z_1) is a point on the cubic.

Hence finally,

$$x_1{}^2x_3 : y_1{}^2y_3 : z_1{}^2z_3=\mu^3-\nu^3 : \nu^3-\lambda^3 : \lambda^3-\mu^3$$

$$=x_1{}^3(y_1{}^3-z_1{}^3) : y_1{}^3(z_1{}^3-x_1{}^3) : z_1{}^3(x_1{}^3-y_1{}^3)$$

$$\therefore \quad x_3 : y_3 : z_3=x_1(y_1{}^3-z_1{}^3) : y_1(z_1{}^3-x_1{}^3) : z_1(x_1{}^3-y_1{}^3)$$

i.e., the co-ordinates of the tangential are proportional to

$$x_1(y_1{}^3-z_1{}^3) : y_1(z_1{}^3-x_1{}^3) : z_1(x_1{}^3-y_1{}^3).$$

If all the three points A, B, C coincide, we have

$$x_1{}^3+y_1{}^3+z_1{}^3=0$$

and consequently $\qquad x_1y_1z_1=0.$

Therefore, at a point of inflexion, we must have

$$x_1{}^3+y_1{}^3+z_1{}^3=0, \qquad x_1y_1z_1=0.$$

It is to be noticed that

$$\frac{x_3{}^3+y_3{}^3+z_3{}^2}{x_3y_3z_3}=\frac{x_1{}^3+y_1{}^3+z_1{}^3}{x_1y_1z_1}.$$

51. PENCIL OF TANGENTS:

THEOREM: *The four tangents which can be drawn from any point of a cubic form a pencil of constant cross-ratio.**

This theorem is due to Salmon and the following geometrical proof is given by him.

The polar conic of a point O on the cubic passes through the points of contact A, B, C, D of the tangents drawn from O, and it touches the cubic at the point O, *i.e.*, the polar conic of O passes through a consecutive point O′, and the points of contact A, B, C, D.

Again, the four tangents from O′ are indefinitely near to OA, OB, OC, OD, and intersect them at their points of contact A, B, C, D. But, the six points O, O′, A, B, C, D lie on a conic, and therefore O(ABCD)=O′(ABCD). Thus we see that this ratio remains the same as we pass from any point of the curve to the consecutive one. Hence it is constant for all points of the cubic.

52. ANALYTICAL PROOF †:

Let the tangents at any two points A and B on the cubic intersect at C. Taking ABC for the triangle of reference, the equation of the curve can be put into the form:

$$z^3+z^2(ax+by)+xy(lx+my+nz)=0$$

The line $y=\lambda z$ meets the cubic, where

$$z^2(1+b\lambda)+(m\lambda^2+n\lambda+a)zx+l\lambda x^2=0$$

Hence, it is a tangent, *i.e.*, meets the curve in two consecutive points, if

$$(m\lambda^2+n\lambda+a)^2=4l\lambda(1+b\lambda)$$

* Salmon, Crelle, Bd. 42 (1351), p. 274.

† This proof is taken from Prof. Hilton's Plane Algebraic Curves, Chap. XIV, §4. An elementary proof was given by W. P. Milne, Quarterly Journal of Math., Vol. 42 (1911), p. 251 and a projective proof, considering the cubic as the projection of the curve of intersection of two conicoids, was given by Prof. A. C. Dixon, Messenger of Math., Vol. 26 (2) (1896-97), p. 53.

whence the equation of the tangents from A is—

$$(my^2+nyz+az^2)^2=4lyz^2(z+by) \qquad \dots \quad (1)$$

Now, the cross-ratio of this pencil is given by

$$I^3(\sigma+1)^2(\sigma-2)^2(\sigma-\tfrac{1}{2})^2=27J^2(\sigma^2-\sigma+1)^3 *$$

where I and J are the two invariants of the quartic (1) and can be calculated in terms of the co-efficients in (1).

Since the expressions for I and J are found to be symmetrical in a, b, and l, m, it follows that the tangents from B have the same cross-ratio, which establishes the theorem.

From known properties of cross-ratio of a pencil, it is easily established that the sixteen points of intersection of two pencils of four tangents, which can be drawn from two points of a cubic lie, four by four, on four conics, which pass through the vertices of the pencils.† This is known as Hart's Theorem.

For a real curve, the cross-ratio is real, if the four tangents are all real, or all imaginary (conjugate in pairs). It will be a complex quantity of modulus 1, or the real quantity -1, if of the four tangents two are real and two conjugate imaginary.

53. Cross-ratio in terms of the parameter:

From what has been said above it is evident that each cubic is characterised by a definite value of this cross-ratio, which is an absolute invariant for any projective transformation, and this is the single absolute invariant of the curve.‡

If the cubic be given in the canonical form

$$x^3+y^3+z^3+6mxyz=0$$

the cross-ratio must be a definite function of the parameter m.

* Burnside and Panton, Theory of Equations, Vol. I, Ex. 16, p. 148-50

† Salmon, Crelle, Bd. 42 (1851), p. 275.

‡ J. Thomæ, Über orthogonale Invarianten der Kurven dritter Ordnung (Leipziger, Ber. 51, 1899).

Now, the equation of the four tangents drawn from any point of the cubic

$$f \equiv x^3+y^3+z^3+6mxyz=0$$

can be easily obtained. Thus the equation of the four tangents drawn from the point (0, 1, −1) is given by *

$$P \equiv (\Delta f)^2-4f.\Delta' f=0,$$

where $$\Delta f=\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)(x^3+y^3+z^3+6mxyz)$$

$$=3(y-z)(y+z-2mx)$$

and $$\Delta' f=3(y+z-2mx).$$

$$\therefore \quad P \equiv \{3(y-z)(y+z-mx)\}^2-4.3(y+z-2mx)\times f$$

$$=3(y+z-2mx)\{3(y-z)^2(y+z-2mx)$$
$$-4(x^3+y^3+z^3+6mxyz)\}=0.$$

To determine the cross-ratio of this pencil, consider the section by the line $z=0$.

Then $$(2mx-y)(4x^3+y^3+6mxy^2)=0$$

i.e. $$8mx^4-4x^3y+12m^2x^2y^2-4mxy^3-y^4=0 \quad \dots \quad (3)$$

whence $$I=8m(-1)-4(-1)(-m)+3(2m^2)^2$$
$$=12m(m^3-1)$$

and $$J=8m.\ 2m^2.\ (-1)+2(-1)(2m^2)(-m)$$
$$-8m(-m)^2-(-1)(-1)^2-(2m^2)^3$$
$$=-8m^6-20m^3+1$$

Thus the equation giving the cross-ratio becomes

$$\frac{16m^3(m^3-1)^3}{(8m^6+20m^3-1)^2}=\frac{(\sigma^2-\sigma+1)^3}{(\sigma+1)^2(\sigma-2)^2(2\sigma-1)^2} \quad \dots \quad (4)$$

This gives twelve values of m for a given value of σ, and consequently in the system of cubics, there are twelve curves which have a given cross-ratio σ.

* Theory of Plane Curves, Vol. I, § 67.

54. HARMONIC AND EQUIANHARMONIC CUBICS: *

If σ is a root of $\sigma^2-\sigma+1=0$, *i.e.*, when σ is equal to $-\omega$ or $-\omega^2$, where ω and ω^2 are the imaginary cube roots of unity, the ratios become equal, three by three, and are said to be *equianharmonic.* They give four values of m and the four curves corresponding to these values are said to be *equianharmonic.*

If σ is a root of any one of the equations $\sigma+1=0$, $\sigma-2=0$, $2\sigma-1=0$, there are only six different values of m, and the curves coincide two at a time; *i.e.*, if $\sigma=-1$, or $1/\sigma=-1$, then the remaining four values become respectively equal to 2 and $\frac{1}{2}$, two by two. This ratio is said to be *harmonic* and the curves corresponding to the six values of m are said to be *harmonic.*

Thus, *corresponding to each value of σ, there are twelve different cubics, and of these six are harmonic and four equianharmonic.*

For all equianharmonic curves, $\sigma^2-\sigma+1=0$, and consequently $I=0$, and this gives two real values of m.

For all harmonic curves $J=0$, and there are only two real values of m.

If $\sigma=1$, 0 or ∞, then the six ratios coincide in pairs and they are respectively equal to 1, 0, ∞. If the curve has a double point, two of the four tangents must coincide and the discriminant of the equation (3) must vanish. This requires that $I^3-27J^2=0$. The expression for this in terms of m is $(1+8m^3)^3=0$.

Hence the condition for a double point is $1+8m^3=0$.

We obtain the degenerate cubic consisting of the sides of the fundamental triangle, when $m=\infty$.

It can be easily seen that a harmonic cubic is the Hessian of its Hessian, and the Hessian of an equianharmonic cubic degenerates.

* Aronhold, Crell, Bd. 39 (1850), p. 153.

Ex. 1. Find the cross-ratio of the pencil of tangents drawn from any point on the cubic

$$(x+y+z)^3 = kxyz.$$

Ex. 2. If all the polar conics for a cubic have a common self-conjugate triangle, the cubic is equianharmonic.

Ex. 3. The polar conics of a point P of the plane *w.r.t.* a pencil of cubics form a pencil, whose base-points are the points of contact of the four tangents drawn from P to the curve passing through P (Cayley).

Ex. 4. Establish Salmon's theorem on the cross-ratio of the pencil of tangents of a cubic, by using the form

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3.$$

[Here $\mathrm{I} = 16 g_2$, $\mathrm{J} = 64 g_3$.]

Ex. 5. Show that for a harmonic cubic $g_2 = 0$, and for an equianharmonic cubic $g_3 = 0$.

55. Conjugate Poles :

THEOREM: *If the polar conic of a point A with respect to a cubic breaks up into two right lines intersecting at B, the polar conic of B breaks up into two right lines intersecting at A.*

The two points A and B are called *Conjugate Poles.**

If (x_1, y_1, z_1) and (x_2, y_2, z_2) be the co-ordinates of A and B respectively, the polar conic of A *w.r.t.* the cubic

$$f \equiv x^3 + y^3 + z^3 + 6mxyz = 0 \quad \text{is :—}$$

$$\phi \equiv x_1(x^2 + 2myz) + y_1(y^2 + mzx)$$
$$+ z_1(z^2 + 2mxy) = 0.$$

If this has a double point at B (x_2, y_2, z_2), we must have

$$\partial \phi / \partial x_2 = 0, \qquad \partial \phi / \partial y_2 = 0,$$
$$\partial \phi / \partial z_2 = 0.$$

* Cayley, "*A Memoir on Curves of the Third Order,*"—Coll. Papers, Vol. II, p. 382.

These conditions reduce to the three equations:

$$\left.\begin{aligned} x_1x_2+m(y_1z_2+y_2z_1)=0 \\ y_1y_2+m(z_1x_2+z_2x_1)=0 \\ z_1z_2+m(x_1y_2+x_2y_1)=0 \end{aligned}\right\} \qquad \dots \quad (1)$$

Since these equations are symmetrical in the co-ordinates of the two points, it follows that the polar conic of B also breaks up into two right lines intersecting at A.

By eliminating (x_1, y_1, z_1) or (x_2, y_2, z_2) between the equations (1), we obtain the same determinant

$$\begin{vmatrix} x_1 & mz_1 & my_1 \\ mz_1 & y_1 & mx_1 \\ my_1 & mx_1 & z_1 \end{vmatrix} = 0$$

i.e., $\quad m^2(x_1^3+y_1^3+z_1^3)-(1+2m^3)x_1y_1z_1=0.$

This shows that the locus of both the points is the Hessian, and it follows in consequence that the Steinerian of a cubic coincides with its Hessian.

56. Net of polar conics:

The equation of the polar conic of a point (x', y', z') is

$$x'\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+z'\frac{\partial f}{\partial z}=0 \qquad \dots \quad (1)$$

Thus the system of polar conics of a cubic curve representing the doubly infinite system of first polars forms a *net*, if (x', y', z') are regarded as parameters.*

* In general, a net of curves of order m may be considered as a system of first polars of a curve of order $(m+1)$. Any three curves of the net generally contain $\frac{3}{2}m(m+3)$ constants. The first polars of three different points involve $\frac{1}{2}(m+1)(m+4)$ constants of the original curve and the six co-ordinates of the three points, and therefore, in all $\frac{1}{2}(m+1)(m+4)+6$ constants. These two numbers will be equal, if $m=2$. Hence, a net of conics can always be considered as the system of first polars of a curve of the third order.

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It should be remembered that so long as we deal with a pencil of conics, each point A corresponds to the conjugate point B, through which pass the polars of A with respect to the same pencil. But in the case of a net, this is not, in general, the case. There is a series of points A whose polars with respect to three, and therefore all, conics of the net pass through one and the same point B. The locus of A is called the "*Jacobian*" of the net of conics. The points B also lie on the same curve.

The polar lines of A (x_1, y_1, z_1) *w.r.t.* the three conics

$$f_x=0, \qquad f_y=0, \qquad f_z=0$$

will be concurrent at B (x_2, y_2, z_2)

if
$$\left.\begin{aligned} x_1x_2+m(z_1y_2+z_2y_1)&=0\\ y_1y_2+m(z_1x_2+z_2x_1)&=0\\ z_1z_2+m(x_1y_2+x_2y_1)&=0 \end{aligned}\right\} \qquad \dots \quad (1)$$

which shows that A and B are points on the Hessian. But they are conjugate points with respect to each of the conics $f_x=0$, $f_y=0$, $f_z=0$; and consequently with respect to all conics of the system

$$x'f_x+y'f_y+z'f_z=0.*$$

Hence *the Hessian of a cubic is the Jacobian of the net of polar conics*

$$x'f_x+y'f_y+z'f_z=0.$$

The line AB is cut in involution by the system of polar conics, A and B being the foci of involution.

If
$$lf_x+mf_y+nf_z=0$$

breaks up into two right lines, they intersect at A or B and touch the Cayleyan.

* Salmon, Conics, § 388.

57. THEOREM.

The polar line with respect to the cubic of a point A on the Hessian touches this latter curve at the conjugate pole B.

The polar line of a point A (x_1, y_1, z_1) *w.r.t.* the cubic

$$f \equiv x^3+y^3+z^3+6mxyz=0$$

is

$$x(x_1^2+2my_1z_1)+y(y_1^2+2mz_1x_1)$$
$$+z(z_1^2+2mx_1y_1)=0 \quad \ldots \quad (1)$$

From the first and third of the equations in the preceding article we obtain—

$$\frac{x_1}{mz_2^2-m^2x_2y_2}=\frac{y_1}{m^2y_2^2-z_2x_2}=\frac{z_1}{mx_2^2-m^2y_2z_2}=k \text{ (say)}$$

Then, $\quad x_1^2+2my_1z_1=k^2[(mz_2^2-m^2x_2y_2)^2$
$$+2m(m^2y_2^2-z_2x_2)(mx_2^2-m^2y_2z_2)]$$
$$=k^2[3m^4x_2^2y_2^2+m^2z_2(z_2^3-2m^3y_2^3-2x_2^3)]$$

But (x_2, y_2, z_2) being a point on the Hessian,

$$m^2(x_2^3+y_2^3+z_2^3)-(1+2m^3)x_2y_2z_2=0.$$

$$\therefore \quad m^2z_2^3=(1+2m^3)x_2y_2z_2-m^2(x_2^3+y_2^3)$$

$$\therefore \quad x_1^2+2my_1z_1=k^2\,[3m^4x_2^2y_2^2-3m^2x_2^3z_2$$
$$+(1+2m^3)(x_2y_2z_2^2-m^2y_2^2z_2)]$$
$$=k^2(m^2y_2^2-z_2x_2)\,[3m^2x_2^2-(1+2m^3)y_2z_2].$$

Similarly,

$$y_1^2+2mz_1x_1=k^2(m^2y_2^2-z_2x_2)\,[3m^2y_2^2-(1+2m^3)z_2x_2]$$

and

$$z_1^2+2mx_1y_1=k^2(m^2y_2-z_2x_2)\,[3m^2z_2^2-(1+2m^3)x_2y_2]$$

$\therefore$ The equation (1) of the polar line becomes

$$x[3m^2x_2^2-(1+2m^3)y_2z_2]+y[3m^2y_2^2-(1+2m^3)z_2x_2]$$
$$+z[3m^2z_2^2-(1+2m^3)x_2y_2]=0$$

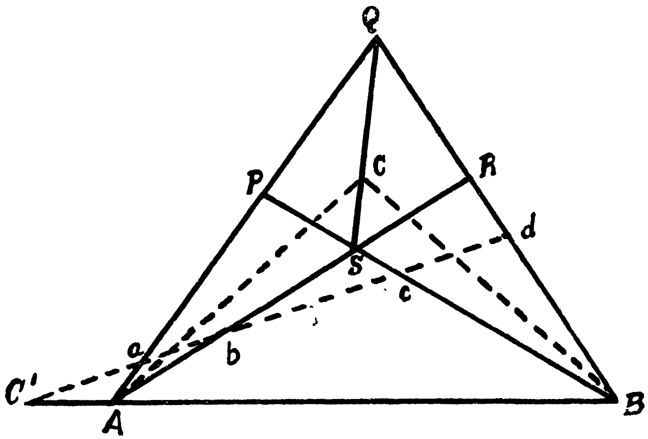
which is the equation of the tangent to the Hessian at the point B (x_2, y_2, z_2).

A simple geometrical proof may also be given by considering the Hessian as the envelope of lines two of whose poles, with respect to the cubic, coincide, when the tangent at any point on the Hessian becomes the polar line of the conjugate point. (See Art. 27.)

58. THEOREM :

*The tangents to the Hessian at two conjugate poles A, B intersect on the Hessian at a point C, which is conjugate to the third point in which AB intersects the Hessian again.**

Let P, Q, R, S be the four points of intersection of the degenerate polar conics of A and B.



The lines PR and QS then may be taken to constitute the polar conic of some point C′ on AB, which again must be a point on the Hessian. If they intersect at C, then C is a point on the Hessian conjugate to C′.

* Cremona, *Introduzioni ad una teoria, etc.*, § 133.

Again, the polar line of A with respect to the cubic is also the polar line with respect to the polar conic of A, *i.e.*, *w.r.t.* the conic consisting of PS, QR. But the triangle ABC is self-polar for all conics through P, Q, R, S. Hence BC is the polar line of A which is consequently the tangent to the Hessian at B. Similarly, AC is the tangent at A. Hence the tangent at A and B intersect at C on the Hessian, which is the point conjugate to C′.

Thus, *two conjugate points have the same tangential on the Hessian.*

Since the polar conic of C passes through A and B, it consists of the line AB and another line passing through C′, the third point where AB meets the Hessian. Thus we obtain the following theorem :—

A line which joins two conjugate poles on the Hessian forms a part of the polar conic of the point conjugate to the third point, where the line intersects the Hessian.

From what has been said, it is seen that the conjugate point B is the point of contact of a tangent drawn from the tangential C of A. But three other tangents, different from the tangent at A, can be drawn from C to the curve, and the points of contact B, B′, B″ are conjugate to A with respect to three different cubics, for which the given cubic $H=0$ may be taken as the Hessian. Hence the points of a cubic can be arranged into conjugate pairs in three different ways.

59. SCHRÖTER'S METHOD OF CONSTRUCTION OF THE CUBIC.

If we start with any pair of conjugate points P and P′, with a common tangential point T, and join P, P′ to any point Q on the curve, then PQ and P′Q intersect the cubic again in the points R and R′ respectively. The lines P′R and PR′ intersect the curve at the same point Q′, conjugate to Q. For, let P_1 and P_1' be two other points on the curve consecutive to P and P′ respectively.

Then P_1PRP_1' $P'R'$ is a hexagon inscribed in the cubic. Two pairs of opposite sides (P_1P, $P_1'P'$) and (PR, $P'R'$) intersect respectively in the points T and Q on the curve. The third pair $P_1'R$ and P_1R' (which ultimately coincide with $P'R$ and PR' respectively) intersect at the point Q′ on the curve (§15).

Again, RQR′Q′ is a quadrilateral inscribed in the cubic, whose opposite sides intersect at the points P and P′ on the curve. Therefore the opposite vertices Q, Q′ and R,R′ have the same tangential points, or in other words, Q and Q′ are conjugate poles, as also are R and R′. Hence we obtain the theorem :—

On each cubic there are three different systems of conjugate points such that any two pairs joined cross-wise give another pair of conjugate points.

Thus a sufficient number of points of a cubic can easily be determined, when three pairs of conjugate points are given. This method, in fact, is used by Schröter* for generating a cubic curve

Schröter † has studied the properties of cubic curves with reference to conjugate points and gives the following general theorem for finding conjugate points on cubic curves.

The four common tangents to each pair of three conics not belonging to a pencil form a complete quadrilateral. The eighteen vertices of the three quadrilaterals thus formed lie on a curve of order three and form nine pairs of conjugate points.

Ex. 1. If three polar conics of a cubic are rectangular hyperbolas, prove that every polar conic is a rectangular hyperbola.

Ex. 2. In *Ex.* 1 shew that the Hessian is a cubic through the circular points at infinity whose singular focus lies on the Hessian.

Ex. 3. Shew that the co-ordinates of two conjugate points A and B on the curve $z(x^2+y^2)=3y(y^2-3x^2)$ may be taken as

$$(\cos\alpha,\ \sin\alpha,\ -3\sin 3\alpha) \quad \text{and} \quad (-\cos\alpha,\ \sin\alpha,\ -3\cos 3\alpha)$$

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* Schröter, Math. Ann. Bd. 5 (1872), p. 50.

† Schröter, Die Theorie der ebenen Kurven 3 Ordnung (1888), p. 35.

60. THE CAYLEYAN :

DEFINITION : The envelope of the line joining two conjugate poles on the Hessian was called by Prof. Cayley the *Pippian* ; but it is called by Cremona the *Cayleyan*.*

From § 56 it follows that the Cayleyan is the envelope of lines which are cut in involution by the system of polar conics, and it is the envelope of the lines which constitute the polar conics of points on the Hessian.†

The equation of the polar conic of a point (x', y', z') with respect to a cubic $f=0$ may be written as

$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=0 \qquad \ldots \quad (1)$$

where a', b', c', ...are the second differential co-efficients of f, in which x, y, z, have been replaced by x', y', z' respectively.

If it breaks up into two lines, the equation must be identical with

$$(\lambda x+\mu y+\nu z)\left(\frac{a'x}{\lambda}+\frac{b'y}{\mu}+\frac{c'z}{\nu}\right)=0 \qquad \ldots \quad (2)$$

Comparing the co-efficients of yz, zx and xy in (1) and (2) we obtain—

$$2f'=\frac{b'\nu}{\mu}+\frac{c'\mu}{\nu} \qquad 2g'=\frac{c'\lambda}{\nu}+\frac{a'\nu}{\lambda}$$

$$2h'=\frac{a'\mu}{\lambda}+\frac{b'\lambda}{\mu}. \qquad \ldots \quad (3)$$

Eliminating (x', y', z') between equations (3), the tangential equation of the Cayleyan can be easily obtained.

* Cayley, Journal de Math., Vol. 9(1) (1844), p. 290.

† In the case of curves of higher degrees the envelope of lines joining corresponding points is distinct from the envelope of lines into which the polar conics break up.

Using the canonical form, we have—

$$a'=6x',\quad b'=6y'\quad c'=6z',\quad f'=6mx',\quad g'=6my',\quad h'=6mz'$$

and the equations (3) become—

$$\left.\begin{aligned} x'\mu^2+y'\lambda^2-2m\lambda\mu z'&=0\\ x'\nu^2-2m\nu\lambda y'+\lambda^2 z'&=0\\ -2m\mu\nu x'+\nu^2 y'+\mu^2 z'&=0 \end{aligned}\right\}$$

Eliminating (x', y', z') between these equations, we obtain the equation of the Cayleyan in the form:

$$\begin{vmatrix} \mu^2 & \lambda^2 & -2m\lambda\mu \\ \nu^2 & -2m\lambda\nu & \lambda^2 \\ -2m\mu\nu & \nu^2 & \mu^2 \end{vmatrix}=0$$

or $$C\equiv m(\lambda^3+\mu^3+\nu^3)+(1-4m^3)\lambda\mu\nu=0 \qquad \dots \quad (4)$$

omitting the factor $\lambda\mu\nu$, since it cannot form the part of the envelope, the Cayleyan being of the third class.

61. THEOREM:

The nine harmonic polars of a cubic are cuspidal tangents to the Cayleyan.

Since the inflexional tangent and the corresponding harmonic polar constitute the degenerate polar conic of a point of inflexion, the inflexional tangents and the harmonic polars touch the Cayleyan.

The equation of the Cayleyan may be written as

$$C\equiv\lambda^3+\mu^3+\nu^3+6k\lambda\mu\nu=0, \text{ where } 6k=(1-4m^3)/m$$

which again can be written in the form

$$(\lambda+\mu-2k\nu)(\omega\lambda+\omega^2\mu-2k\nu)(\omega^2\lambda+\omega\mu-2k\nu)$$
$$=-(1+8k^3)\nu^3.$$

showing that the three factors on the left give the cusps on the curve.

For, let P be the point $\lambda+\mu-2k\nu=0$. $\nu^3=0$ represents the third vertex C counted thrice. Therefore the lines joining P to three coincident points at C are tangents to the Cayleyan, *i.e.*, three contiguous tangents pass through P. Hence P is a cusp on the Cayleyan and PC is the cuspidal tangent. Similarly, the other two factors give two other cusps Q and R, QC, RC being the cuspidal tangents.

The line joining

$$\lambda+\mu-2k\nu=0 \text{ and } \nu=0 \text{ gives } \lambda+\mu=0, \; i.e., \; \mu=-\lambda$$

and the point-equation of PC is $x-y=0$. Similarly, the point-equations of QC and RC are respectively $x-\omega y=0$, $x-\omega^2 y=0$; but these are the equations of the harmonic polars of the points of inflexion of the cubic on the line $z=0$. Similarly, it can be shown that all the harmonic polars are cuspidal tangents to the Cayleyan.

Combining this with the theorem of the preceding article, since these equations are independent of m, we may deduce the following theorem :—

The nine inflexional tangents of a curve of the third order in a syygetic system determine a curve of the third class, having the same cuspidal tangents.

Ex. 1. The Cayleyan of a non-singular cubic is a sextic curve with nine cusps.

Ex. 2. The poles of a line *w.r.t.* a cubic are the vertices of a quadrangle whose sides touch the Cayleyan and the diagonal points lie on the Hessian.

Ex. 3. Find the equation of the Cayleyan of the curves .

$$(i) \quad z^2x=y(y-x)(y-k^2x)$$

$$(ii) \; a(x^2\pm y^2)=x^3 \qquad (iii) \; x^3+y^3=3axy$$

Ex. 4. What is the Cayleyan of a cuspidal cubic ?

62. ORDER OF THE CAYLEYAN:

We have seen (§ 27) that the poles of a tangent to the Hessian at any point M are the points of contact with the Cayleyan of the lines which constitute the polar conic of M and the point R counted twice, where R is the point conjugate to the tangential L of M. Thus, the locus of the poles of the tangents to the Hessian includes the Cayleyan and the Hessian counted twice.

The polar line of a point (x', y', z') being

$$x\frac{\partial f}{\partial x'}+y\frac{\partial f}{\partial y'}+z\frac{\partial f}{\partial z'}=0$$

the condition that this should touch the Hessian involves the co-efficients in the sixth degree, and therefore the variables in the twelfth degree. Hence the locus of the poles is of twelfth degree, but contains the Hessian counted twice. Hence the Cayleyan is of degree six.

The equation of the Cayleyan is obtained in the form—

$$x^6+y^6+z^6-2(1+16k^3)(x^3y^3+y^3z^3+z^3x^3)$$
$$-24k^2xyz(x^3+y^3+z^3)-24k(1+2k^3)x^2y^2z^2=0$$

where $$6k\equiv(1-4m^3)/m$$

Ex. 1. Shew that the Cayleyan of a cubic is of class three.

Ex. 2. A line is divided in involution by three given conics. Shew that the envelope of the line is of third class.

Ex. 3. The polar line of A *w.r.t.* the polar conic of the conjugate pole B for any syzygetic cubic touches the Cayleyan.

63. If λ, μ, ν be eliminated between the equations (3) § 60, we obtain the equation of the Hessian. The form of this equation is the same as that of the given cubic, and therefore the Cayleyan C stands to the vertices of the fundamental triangle in the same relation as the original curve stands to the sides. The tangents to this curve drawn from the vertices correspond to the inflexional tangents on the original curve. If m is regarded as a parameter, the equation (4) represents the system of Cayleyan curves of the system of cubics defined by the canonical form.

64. THE POLE CONIC OF A GIVEN LINE* :

DEFINITION: The locus of points whose polar conics with regard to a given cubic touch a given line is a conic and is called the *Pole Conic* of the line.

The polar conic of a point (x', y', z') with regard to a cubic may be written as—

$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=0$$

where a', b', c', etc , have the significance of § 60.

The condition that this touches the given line

$$lx+my+nz=0$$

is—

$$A'l^2+B'm^2+C'n^2+2F'mn+2G'nl+2H'lm=0 \qquad \dots \quad (1)$$

where A', B', C',......are the minors of a', b', c',......in the determinant

$$\begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}$$

and are therefore functions of second degree in (x', y', z'). Hence the locus of (x', y', z') is a conic, which is called the *Pole Conic* of the given line and its equation is—

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0 \qquad \dots \quad (2)$$

Ex. 1. Shew that the pole conic of a line can be defined as—

(*i*) the envelope of polar lines of points on the line,

(*ii*) the locus of poles of the line *w.r.t.* the polar conics of all points on the line.

Ex. 2. The tangents to a cubic touch their pole conics.

Ex. 3. Deduce the tangential equation of the cubic.

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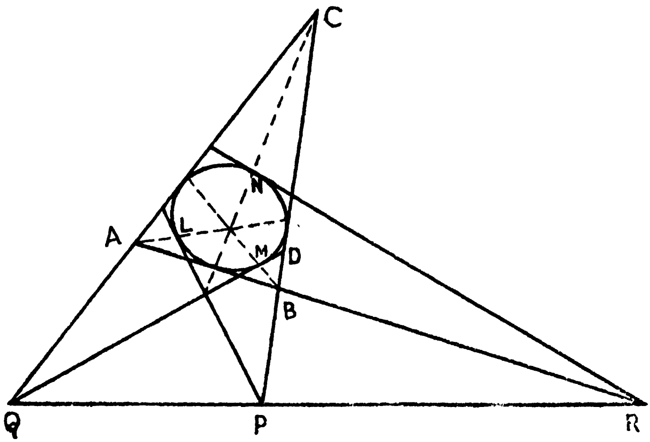
* Theory of Plane Curves, Vol. I, §77.

Ex. 4. The pole conic of a tangent to the Cayleyan reduces to a point. [The tangent together with its conjugate line constitutes the polar conic of some point, and all the polar lines of points on the tangent pass through this point.]

Ex. 5. The pole conic of the line at infinity is the locus of points whose polar conics touch the line at infinity, *i.e*, become parabolas. (See Vol. I, § 91.)

65. Properties of the Pole Conic *:

From Maclaurin's theorem, it follows that if a line intersects a cubic in the points P, Q, R and tangents be drawn at these points, the polar line of any point on the



line with respect to the cubic is the same as its polar line with respect to the three tangents. Now, if these tangents be $xyz=0$, the polar line of (x', y', z') is—

$$xy'z'+yz'x'+zx'y'=0 \quad \dots \quad (1)$$

Also $$lx'+my'+nz'=0 \quad \dots \quad (2)$$

* Salmon, H. P. Curves, § 185.

Hence, the pole conic of the line is the envelope of (1), subject to the condition (2),

$$i.e.,\qquad \sqrt{(lx)} + \sqrt{(my)} + \sqrt{(nz)} = 0 \qquad \ldots \quad (3)$$

which is an inscribed conic of the triangle formed by the tangents, D, E, F being the points of contact. But from each of P, Q, R a second tangent can be drawn to the conic, and let L, M, N be the points of contact of these tangents respectively. By the properties of the inscribed conic, the groups A, L, D ; B, M, E ; and C, N, F are each collinear and these lines meet in a point. By the harmonic properties of quadrilaterals, the points D, E, F are harmonic conjugates of P, Q, R with respect to the point-pairs B, C ; C, A and A, B respectively.

66. Intersections of a line with its Pole Conic :

The pole conic of the line $x=0$ *w.r.t.* the cubic

$$x^3+y^3+z^3+6mxyz=0$$

is $yz=m^2x^2$, having $y=0$ and $z=0$ as tangents at the points where $x=0$ meets it.

The lines joining the vertex A with the points where $x=0$ meets the cubic are—

$$y+z=0, \quad y+\omega z=0, \quad y+\omega^2 z=0 \qquad \ldots \quad (1)$$

These lines together with $y=0$ form an equianharmonic pencil, since the six cross-ratios of this pencil are

$$-\omega, -\omega, -\omega ; -\omega^2, -\omega^2, -\omega^2.$$

Similarly, $z=0$ forms with the lines (1) an equianharmonic pencil. Therefore the transversal $x=0$ meets both these pencils in equianharmonic ranges. We thus obtain the theorem :

The points of intersection of a line with its pole conic are situated equianharmonically with regard to the points where the line intersects the original curve.

67. THEOREM :

The polar conics (line-pairs) of two conjugate poles touch the Cayleyan in four collinear points.

Let AQ, AS and BQ, BS be the polar conics of two conjugate poles B and A respectively (Fig. § 58). Let C′ be the point conjugate to C. The polar conic of C consists of AB and a line C′*abcd* drawn through C′. Let this line intersect AQ, AS, BS, BQ in the points *a*, *b*, *c*, *d* respectively. Then *a*, *b*, *c*, *d* will be the points of contact with the Cayleyan of those lines respectively.

Now the poles of AC are the intersections of the lines AB and C′*abcd* with the lines BQ, BS, *i.e.*, the poles are the points *c*, *d* and the point B counted twice. But AC is the tangent to the Hessian at A. Therefore the points *c*, *d* are the points of contact of BS and BQ with the Cayleyan. Similarly, *a* and *b* are the points of contact of AQ and AS with the Cayleyan, which proves the proposition.

68. THEOREM :

*The line joining two conjugate poles is divided harmonically at the point where it cuts the Hessian again and its point of contact with the Cayleyan.**

Let C′ be the complementary point † on AB, and join the two conjugate poles C and C′. If D′ be the complementary point on CC′, the line CC′ and the conjugate line through D′ constitute the polar conic of some point on the Hessian. This conjugate line DD′ intersects AB in its point of contact with the Cayleyan.

But every polar conic divides harmonically the line joining two conjugate poles. Hence the lines CC′ and DD′ divide AB harmonically, which proves the theorem.

* Cayley, Phil. Trans., Vol. 147 (1857), p. 425.

† The point in which the line joining two conjugate poles meets the Hessian again is called by Salmon the *complementary point*. It is the point where the line is cut by the conjugate line, the two forming the polar conic of some point.—See H. P. Curves, § 176.

69. THEOREM :

The Hessian touches the nine inflexional tangents at their intersections with the corresponding harmonic polars.

The polar conic of a point of inflexion A consists of the inflexional tangent and the corresponding harmonic polar intersecting each other at B, which is conjugate to A, and both A and B lie on the Hessian. The lines are tangents to the Cayleyan. The inflexional tangent AB is the polar line of A, and therefore is the tangent to the Hessian at the conjugate pole B. Hence the inflexional tangent AB touches the Hessian at its intersection with the corresponding harmonic polar.

Ex. The three tangents which can be drawn to the Cayleyan from any point on the Hessian are the line joining the point with the conjugate pole and the two lines constituting the polar conic of the conjugate pole (Cremona, § 133).

70. POINTS OF CONTACT OF THE CAYLEYAN WITH THE HESSIAN :

The inflexional tangent AB is an ordinary tangent to the Cayleyan, while the harmonic polar through B is a cuspidal tangent (§ 61). But AB is also a tangent to the Hessian at B and therefore the complementary point is B, and the conjugate line to AB, on which the point of contact with the Cayleyan must lie, is the harmonic polar, intersecting AB at B. Hence B is the point of contact of AB with the Cayleyan, *i.e.*, the point of contact of the inflexional tangent with the Cayleyan is the point where it meets the harmonic polar. Thus the Hessian and the Cayleyan touch each other at B with the common tangent AB, which is the inflexional tangent of the cubic at the conjugate pole A. Thus we obtain the theorem :—

The Cayleyan and the Hessian touch each other at nine points, which are the points of intersection of the inflexional

*tangents with the corresponding harmonic polars, the common tangents being the nine inflexional tangents to the cubic.**

That the Hessian and the Cayleyan correspond reciprocally has been shown by Cremona (§ 41). For further information, see Scott, Phil. Trans., Vol. 185 (1894), and Gordan, Trans. Am. Math. Soc., Vol. 1 (1900). Properties of cubic curves with reference to poles and polars were discussed by Loudon, Math. Ann. Bd. 36 (1890).

Ex. 1. Shew that the Cayleyan of a cuspidal cubic degenerates into the cusp and the intersection of the cuspidal and inflexional tangents.

Ex. 2. Shew that the Cayleyan of a nodal cubic is a conic.

Ex. 3. Shew that the Cayleyan is the envelope of the mixed polar line of two points on the Hessian, of which one is the tangential of the other. (Cayley.)

* These two curves cannot touch each other at more than nine points; for the Cayleyan is of degree six and the Hessian of degree three. The number of common points is therefore 18. If they touch each other at more than nine points, they would intersect in more than 18 points, which is impossible. The number of common tangents is also nine, for they can have only 18 common tangents, one being of class 6 and the other of class 3. When they touch at a point, the common tangent at that point counts as two common tangents.

CHAPTER III

Rational Cubics

71. General Properties :

There are three classes of cubic curves with a double point, namely, the nodal cubics, cuspidal cubics and acnodal cubics; and consequently these are the only cubics of deficiency zero, whose equations can be expressed in the parametric forms, *i.e.*, the co-ordinates are expressible as rational and integral functions of a parameter. In the case of a non-singular cubic, the deficiency is unity, and the co-ordinates are expressible in terms of elliptic functions.

Writing the parametric equations of a cubic in the form :—

$$\left.\begin{aligned} x &= a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \equiv f_1(\lambda) \\ y &= b_0\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 \equiv f_2(\lambda) \\ z &= c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 \equiv f_3(\lambda) \end{aligned}\right\} \quad \ldots \quad (1)$$

the implicit equation of the curve may be obtained in a determinant form by eliminating λ between these equations. The resulting equation is of degree three, and thus it represents a curve of the third order, and of class four. It gives the general parametric representation of all the ∞^8 rational curves of the third order in a plane.*

* Igel, Über ebene Kurven dritter ordnung mit einem Doppelpunkt, Math. Ann. Bd. 6 (1873).

72. INFLEXIONS AND DOUBLE POINTS:

If the three points with parameters λ, μ, ν lie on a straight line

$$lx+my+nz=0$$

we must have

$$\left.\begin{aligned} lf_1(\lambda)+mf_2(\lambda)+nf_3(\lambda)&=0\\ lf_1(\mu)+mf_2(\mu)+nf_3(\mu)&=0\\ lf_1(\nu)+mf_2(\nu)+nf_3(\nu)&=0 \end{aligned}\right\} \quad \ldots \quad (1)$$

Eliminating l, m, n, we obtain the determinant equation:

$$\begin{vmatrix} f_1(\lambda) & f_2(\lambda) & f_3(\lambda) \\ f_1(\mu) & f_2(\mu) & f_3(\mu) \\ f_1(\nu) & f_2(\nu) & f_3(\nu) \end{vmatrix} =0$$

which is the product of the two matrices—

$$\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{Vmatrix} \cdot \begin{Vmatrix} \lambda^3 & \lambda^2 & \lambda & 1 \\ \mu^3 & \mu^2 & \mu & 1 \\ \nu^3 & \nu^2 & \nu & 1 \end{Vmatrix}$$

whence we get

$$\begin{vmatrix} 1 & -(\lambda+\mu+\nu) & (\lambda\mu+\mu\nu+\nu\lambda) & -\lambda\mu\nu \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{vmatrix} =0$$

This may be written in the form—

$$k_0+k_1(\lambda+\mu+\nu)+k_2(\lambda\mu+\mu\nu+\nu\lambda)+k_3\lambda\mu\nu=0 \quad \ldots \quad (2)$$

where k_0, k_1, k_2...denote the determinants of the first matrix above written.

If now $\lambda=\mu=\nu$, the parameters of the points of inflexion are obtained from (2) in the form—

$$k_0+3k_1\lambda+3k_2\lambda^2+k_3\lambda^3=0 \qquad \dots \quad (3)$$

which shows that the three inflexions lie on a right line.

The discriminant Δ of (3) is given by—

$$\Delta\equiv(k_0k_3-k_1k_2)^2-4(k_1k_3-k_2{}^2)(k_0k_2-k_1{}^2) \quad \dots \quad (4)$$

and one or all the three inflexions are real, according as Δ is positive or negative.

Again, the double point on the cubic corresponds to two different parameters a and β, and if λ be the parameter of any point, then the three parameters a, β, λ must satisfy the condition (2) identically, *i.e.*, for all values of λ,

$$k_0+k_1(a+\beta+\lambda)+k_2(\beta\lambda+a\lambda+a\beta)+k_3\lambda a\beta=0$$

whence,

$$k_0+k_1(a+\beta)+k_2a\beta=0$$

$$k_1+k_2(a+\beta)+k_3a\beta=0$$

and consequently, a, β are the roots of the equation

$$\begin{vmatrix} 1 & -\rho & \rho^2 \\ k_0 & k_1 & k_2 \\ k_1 & k_2 & k_3 \end{vmatrix}=0 \qquad \dots \quad (5)$$

The discriminant of this equation is the same as in (4) and the double point is a node or a conjugate point, according as Δ is positive or negative.

Hence we obtain the theorem—

A unicursal cubic possesses one or three real points of inflexion, according as it has a node or a conjugate point. A cuspidal cubic has always one real inflexion.

* *Cf.* G. Loria, *Spezielle Alg. und tranz. ebenen Kurven*, Vol. 1, §18.

73. APPLICATION TO CANONICAL FORMS:

The equation of an autotomic cubic can be put into the form—

$$(y^2-kx^2)z=x^3 \quad \dots \quad (1)$$

where the vertex C is a node, a cusp or a conjugate point, according as $k >, =,$ or < 0.

By putting $x=\lambda y$, we can express the co-ordinates of any point in the form—

$$x : y : z=\lambda(1-k\lambda^2) : (1-k\lambda^2) : \lambda^3 \quad \dots \quad (2)$$

The equation of the line joining any two points with parameters α and β may be written as—

$$x\{\alpha^2+\alpha\beta+\beta^2-k\alpha^2\beta^2\}-y\{\alpha\beta(\alpha+\beta)\}$$
$$-z\{(1-k\alpha^2)(1-k\beta^2)\}=0$$

whence the tangent at any point $\alpha=\beta$ is—

$$\alpha^2 x(3-k\alpha^2)-2\alpha^3 y-z(1-k\alpha^2)^2=0 \quad \dots \quad (3)$$

which is of order four in α, and hence the curve is of class four. The parameters of the points of contact are given by (3), and if $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the parameters, we have

$$\Sigma\alpha_1\alpha_2\alpha_3=0, \text{ i.e., } \Sigma\frac{1}{\alpha}=0.$$

By a proper choice of co-ordinates, the equation (1) can be put into the simpler form—

$$(x^2\pm y^2)z=x^3$$

the upper sign belonging to the case of the acnodal, and the lower to that of the crunodal cubic.

The co-ordinates of any point can then be taken proportional to

$$(1\pm\theta^2) : \theta(1\pm\theta^2) : 1$$

Substituting these values in the equation of an arbitrary line $lx+my+nz=0$, the parameters of the three points where the line meets the cubic are given by—

$$(l+n)+m\theta\pm l\theta^2\pm m\theta^3=0$$

If θ_1, θ_2, θ_3 are the parameters, we have—

$$\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1=\pm 1$$

When the line touches the curve at a point of inflexion,

$$\theta_1=\theta_2=\theta_3, \quad \text{whence} \quad \theta^2=\pm\tfrac{1}{3}.$$

Thus we obtain the theorem :

An acnodal cubic has three real inflexions, and a crunodal cubic one real and two imaginary ones.

which was otherwise established in the preceding article.

Ex. 1. Express rationally in terms of a parameter the co-ordinates of any point on the curves :

(*i*) $(x+y)(x-y)^2=4xyz$ (*ii*) $zu_2=u_3$

Ex. 2. Find the locus of a point P such that the points of contact of the four tangents drawn from it subtend a pencil of constant cross-ratio at the node of a nodal cubic

Ex. 3. Express rationally in terms of a parameter the co-ordinates of any point on the cubic

$$a(x-y)^2z=x^3,$$

and shew that the sum of the abscissae of the feet of concurrent normals is constant.

Ex. 4. Shew that the equation of a cubic having one real inflexion at infinity and two at the circular points at infinity may be put into the form—

$$27m(x^2+y^2)=(4m-x)^3$$

Ex. 5. Shew that the co-ordinates of any point on the curve

$$x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0$$

may be taken as $\theta^3 : (1-\theta)^3 : -1.$

74. UNIPARTITE CUBICS:

From equation (1) of the last article, it follows that the parameter for the double point is given by $\lambda = \pm\sqrt{1/k}$.

If the curve has a node, $k > 0$, and λ has two distinct real values; and consequently the curve consists of two branches, of which one corresponds to the values of the parameter between $+\sqrt{1/k}$ and $-\sqrt{1/k}$ through 0, and the other between $+\sqrt{1/k}$ and $-\sqrt{1/k}$ through ∞.

As k becomes smaller and smaller, the second branch gradually disappears and finally forms a loop at the double point. If $k=0$, there is a cusp produced by the gradual contraction of the loop, when the parameter passes through ∞.

If k is negative, the double point becomes a conjugate point and it no longer lies on the continuous branch whose points correspond to the real values of the parameters from $-\infty$ to $+\infty$. This acnode is an isolated point that cannot be included in the description of the curve by a real tracing point.*

Hence *a unicursal cubic consists of a single circuit, i.e., it is unipartite.*

But as has been pointed out,† all unipartite cubics are not necessarily unicursal. A crunodal cubic is unicursal and unipartite, all the points on the curve succeed each other in a definite order, and form a single circuit. The curve may be regarded as consisting of a loop and an infinite branch, the two parts of which are separated by the loop. A cuspidal cubic and an acnodal cubic are each of them unicursal and unipartite.

* The curve, due to the existence of the acnode, has, in fact, two imaginary branches, and cannot, therefore, strictly speaking, be termed *unicursal.*

† Theory of Plane Curves, Vol. I, § 241.

75. BIPARTITE CURVES:

If, however, the curve consists of an oval and an infinite branch, every right line, meeting the oval once, must meet it a second time and no more. Therefore, that line can meet the infinite branch only once. Consequently, no tangent can meet the oval again, and no points of inflexion can lie on the oval part. But from any point outside the oval, two tangents can be drawn to it. Hence the oval is a continuous series of points, from none of which any real tangent, distinct from the tangent at the point, can be drawn to the curve. The four tangents drawn from any point on the infinite branch are all real, two to the oval and two to the branch itself, while the tangents from any point on the oval are all imaginary. The tangent at any point of the infinite branch must meet that branch again and *not* the oval, since, if it meets the oval, it must meet it twice.

Again, the points are not arranged in a single circuit. For, if we start with any point on the oval and proceed continuously, we return to the starting point, without passing through any point on the infinite branch. In this case the co-ordinates cannot be rationally expressed in terms of a single parameter. The curve consists of two distinct branches and is called a *bipartite* curve.*

In fact, a non-singular cubic with unit deficiency consists of—

(1) a continuous portion, called the *odd circuit*, approaching the asymptote at its ends and contained between two lines parallel to the asymptote;

(2) a portion called the *even circuit* or oval, contained between two lines parallel to the asymn' .

Thus, cubic curves may be ' .ided into two main divisions—unipartite cubics and bipartite cubics.

* *Cf.* Salmon, H. . Curves, §§ 200, 201.

76. Collinear Points :

If $lx+my+nz=0$ be the equation of a line, meeting the curve in three points with parameters α, β, γ respectively, substituting the values of x, y, z from the equation (1) of § 73, we obtain

$$\lambda^3(n-lk)-\lambda^2 mk+l\lambda+m=0 \qquad \ldots \quad (1)$$

whose roots are α, β, γ and consequently

$$\alpha+\beta+\gamma=mk/(n-lk)$$

$$\alpha\beta\gamma=-m/(n-lk)$$

whence,

$$\alpha+\beta+\gamma+k\alpha\beta\gamma=0 \qquad \ldots \quad (2)$$

which is the relation to be satisfied by the parameters of three collinear points on the cubic.

If $\alpha=\beta=b$ and $\gamma=a$, the condition becomes

$$a+2b+kab^2=0 \qquad \ldots \quad (3)$$

showing that two tangents can be drawn from any point a on the cubic.

If $k=0$, the double point is a cusp and the condition of collinearity (2) becomes

$$\alpha+\beta+\gamma=0$$

If $k=-1$ (say), the double point is an acnode and the condition of collinearity becomes

$$\alpha+\beta+\gamma-\alpha\beta\gamma=0$$

Putting $\alpha=\tan a$. $\beta=\tan b$, $\gamma=\tan c$, it becomes

$$\tan(a+b+c)=0, \text{ i.e., } a+b+c\equiv 0 \pmod{\pi}.$$

Again, the discriminant of (3) is $\Delta\equiv 1-ka^2$. Hence it follows that if the absolute value of a is less than $\sqrt{1/k}$, b is imaginary, and consequently no real tangent can be

drawn to the curve. But in this case the point lies on the oval. Hence, from a point on the oval, no real tangent can be drawn, as we have otherwise shown.

If $a = \pm\sqrt{1/k}$, both the points of contact coincide with the double point. It then gives $b = \mp\sqrt{1/k}$ as the parameter of the cusp. If $k=0$, one of the roots of (3) is infinite, and the loop contracts into the cusp, and from any point on a cuspidal cubic only *one* tangent can be drawn to the curve and the parameter of this point of contact is $b = -\frac{1}{2}a$, the other tangent corresponds to $b=\infty$, which is therefore an asymptote to the curve.

If $a=b$, *i.e.*, if the tangential coincides with the point itself, it is a point of inflexion, and the above equation (3) becomes

$$3a + ka^3 = 0$$

whose roots give the parameters of the points of inflexion. These parameters are, therefore,

$$a_1 = 0, \quad a_2 = +\sqrt{-3/k}, \qquad a_3 = -\sqrt{-3/k}.$$

These three points of inflexion are real only in the case when $k < 0$, *i.e.*, when the curve has an acnode. When $k > 0$, the curve has a node, and there is only *one* real point of inflexion, as was otherwise shown in § 72.

Again,

$$a_1 + a_2 + a_3 = 0 \quad \text{and} \quad a_1 a_2 a_3 = 0$$

$$\therefore \quad a_1 + a_2 + a_3 + k a_1 a_2 a_3 = 0$$

i.e., the three points of inflexion of a unicursal cubic lie on a right line.

Ex. 1. Shew that the product of the parameters of three collinear points on the curve

$$x^3 + y^3 = 3xyz \qquad \text{is} \quad -1.$$

Ex. 2. A chord subtending a right angle at the node of the cubic $x^3 + y^3 = 3axy$ meets the cubic again at a fixed point.

77. NODAL CUBICS:

Consider the line $x=\lambda y$ drawn through the double point C on the cubic $x^3+y^3+6mxyz$. Eliminating x between their equations, we have $y^3(1+\lambda^3)+6m\lambda y^2z=0$, which gives

$$y/z=-6m\lambda/(1+\lambda^3)$$

and consequently
$$\frac{x}{z}=-6m\lambda^2/(1+\lambda^3)$$

Thus, $$x : y : z=-6m\lambda^2 : -6m\lambda : 1+\lambda^3$$

The product of the parameters of three collinear points $\lambda_1, \lambda_2, \lambda_3$ is equal to -1. If $\lambda_1=\lambda_2$, we obtain $\lambda_1{}^2=-1/\lambda_3$ or, $\lambda_3=-1/\lambda_1{}^2$, *i.e.*, the tangential point λ_3 of λ_1 is $-1/\lambda_1{}^2$, and the points of contact of the tangents drawn from the point λ_3 are given by—

$$\lambda_1{}^2=-1/\lambda_3, \quad i.e., \quad \pm\sqrt{1/\lambda_3}.$$

If the line is an inflexional tangent, $\lambda_1=\lambda_2=\lambda_3$, and the condition is $1+\lambda^3=0$, which gives the parameters of the points of inflexion as $-1, -\omega, -\omega^2$.

If the equation of the acnodal cubic be put into the form $z(x^2+y^2)=y(3x^2-y^2)$, the co-ordinates of any point may be expressed as $(\cos\phi, \sin\phi, \sin 3\phi)$, and the collinearity of three points is obtained in the form—

$$\phi_1+\phi_2+\phi_3\equiv 0 \pmod{\pi}.$$

Ex. 1. The parametric representation of the Folium of Descartes $x^3+y^3+3axyz=0$ is given by—

$$x : y : z=-3a\lambda^2 : -3a\lambda : (1+\lambda^3).$$

Ex. 2. Shew that the co-ordinates of any point on the crunodal cubic

$$x(x^2-y^2)=y(3x^2+y^2)$$

may be put in the form $(\cosh\phi, \sinh\phi, \sinh 3\phi)$.

Ex. 3. Prove that the product of the parameters of six points in which a conic meets the cubic $x^3+y^3=6mxyz$ is $+1$.

Ex. 4. Shew that a nodal cubic and its Hessian are in plane perspective, the node and the inflexional line being the vertex and axis of perspective.

78. CUSPIDAL CUBIC:

The co-ordinates of any point on the cuspidal cubic $y^2z=x^3$ can be expressed as

$$x : y : z=\lambda : 1 : \lambda^3.$$

The line

$$lx+my+nz=0$$

meets the cubic in points given by

$$l\lambda+m+n\lambda^3=0.$$

Hence *the sum of the parameters of three collinear points is zero*, and conversely.

Hence the tangential of a point λ is -2λ, and the point of contact of the tangent, drawn from any point λ is $-\frac{1}{2}\lambda$. At a point of inflexion $3\lambda=0$, *i.e.*, $\lambda=0$.

The equation of a line joining any two points λ_1, λ_2 on the curve is—

$$(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)x-\lambda_1\lambda_2(\lambda_1+\lambda_2)y-z=0.$$

For this is satisfied by the co-ordinates of the two points having parameters λ_1 and λ_2. Hence the equation of the tangent at a point $\lambda=\lambda_1\doteq\lambda_2$ is—

$$3\lambda^2x-2\lambda^3y-z=0.$$

If this tangent passes through a fixed point, the above equation in λ gives the values of the parameters of the points of contact of tangents.

Thus if λ_1, λ_2, λ_3 be the parameters, we have—

$$\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1=0, \ i.e., \ \Sigma\frac{1}{\lambda}=0.$$

Hence *the parameters of the points, the tangents at which meet in a point, are connected by the relation*

$$\Sigma\frac{1}{\lambda}=0.$$

79. INTERSECTIONS WITH AN *n-ic*:

The points of intersection of the cubic with an *n-ic* $f(x, y, z)=0$ are given by $f(\lambda, 1, \lambda^3)=0$, in which the co-efficient of λ^{3n-1} is zero, and consequently the sum of the roots is zero. Thus the parameters of the points of intersection of a cubic with an *n-ic* are connected by the relation $\Sigma\lambda=0$.

In particular, if a conic intersects the cubic in six points with parameters $a, b, c, d. e, f$, then

$$a+b+c+d+e+f=0.$$

Again, if g is a point collinear with e and f on the cubic,

$$e+f+g=0.$$

$$\therefore \qquad g=-(e+f)=a+b+c+d.$$

i.e., if a conic through four points on a cubic intersects this latter in two other points, the line joining them passes through a fixed point on the cubic (§ 19).

Similarly, all other theorems in residuation follow in the case of a cuspidal cubic.

Ex. 1. Express in terms of a single parameter the co-ordinates of a point on the curve

$$y^2z=x^2(x-z).$$

[The co-ordinates of a point are given by

$$x : y : z=\lambda(1+\lambda^2) : (1+\lambda^2) : \lambda^3 \text{]}$$

Ex. 2. A conic osculates a cuspidal cubic at two points. Show that the chord of contact passes through the inflexion.

Ex. 3. The chord in Ex. 2 is divided harmonically by the inflexion and the cuspidal tangent.

Ex. 4. The locus of the intersection of tangents to the semi-cubical parabola $ay^2=x^3$ at points subtending a right angle at the cusp is a nodal cubic having double contact with the curve.

80. CUBICS WITH UNIT DEFICIENCY OR NON-SINGULAR CUBICS :

We have seen that all curves of deficiency zero is representible rationally by means of a parameter, while for curves of unit deficiency, it is necessary to introduce elliptic functions in their parametric representation, and we shall now directly demonstrate the procedure in the case of non-singular cubics ; and for this purpose, we shall make use of special forms of equations of a cubic.

The equation of any curve of the third order (deficiency 1) may be written in the form—

$$F \equiv xz^2 - y(x-y)(k^2x-y) = 0 \qquad \ldots \quad (1)$$

which shows that the point (x, y) is an inflexion with $x=0$ as the tangent. Through this point there evidently pass the three lines—

$$y=0, \quad x-y=0, \quad k^2x-y=0$$

and we get $xz^2=0$, showing that these lines are the tangents drawn from the inflexion (x, y), the points of contact lying on $z=0$, *i.e.*, $z=0$ is the harmonic polar.

It follows then that k^2 represents one of the six cross-ratios of the three tangents and the inflexional tangent, *i.e.*, the pencil of lines.

$$x=0, \quad y=0, \quad x-y=0, \quad k^2x-y=0.$$

The equation (1) is evidently satisfied by putting

$$x : y : z = \mu^3 : \mu : \sqrt{(1-\mu^2)(1-k^2\mu^2)}$$

If now we put $\mu = \operatorname{sn} u$, *i.e.*

$$u = \int_0^{\mu} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \qquad \ldots \quad (2)$$

we obtain $\sqrt{1-\mu^2}=\text{cn}u$ and $\sqrt{1-k^2\mu^2}=\text{dn}u$. Hence the co-ordinates are expressed in terms of elliptic functions in the form—

$$x : y : z = \text{sn}^3 u : \text{sn}u : \text{cn}u\,\text{dn}u$$

Thus then to each value of u corresponds a unique point on F, but the converse proposition is not always true, *i.e.*, to each point does not correspond a unique value of the parameter u, owing to the periodic nature of the elliptic functions.

The integral (2) has, in fact, two moduli of periodicity Ω and Ω', defined as being the values of the integral taken on the two oblique sections of the corresponding Riemann surface, and are given by—

$$\Omega=4\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$\Omega'=2\int_0^{\frac{1}{k}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} -2\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

If u is one value of the integral (2), all the other values are obtained from $u+m\omega+m'\omega'$, where m and m' are integers. For, the periods of sn$u$ are Ω and Ω', while the common periods of sn$u$, cn$u$, dn$u$ are Ω and $2\Omega'$*

Thus to all points of the curve corresponds an infinity of values of the argument, which are comprised within the formula

$$u+\frac{m}{2}\Omega+m'\Omega'$$

i.e., we say that the curve has the periods

$$\omega=\frac{1}{2}\Omega, \quad \omega'=\frac{1}{2}\Omega'$$

* Clebsch, Leçons sur la Geometric, Vol. II, p. 357-59.

81. APPLICATIONS:

If u_1, u_2, u_3 be the arguments relative to three collinear points on a cubic, it is proved in works on the theory of elliptic functions * that

$$u_1+u_2+u_3 \equiv 0 \ (\text{mod. } \omega, \omega')\cdot$$

and conversely, three points of the curve lie on a right line, if the sum of the corresponding parameters u_1, u_2, u_3 is equal to a *period.*

If now we put $u_2=u_3$, so that $u_1+2u_2\equiv 0$, we see that the point of contact of the tangent drawn from the point (u_1) has the parameter of the form

$$u_2=\frac{m\omega+m'\omega'-u_1}{2}$$

where m and m' are integers. We may take m and m' equal to zero and -1, because all higher numbers may be obtained by adding integral multiples of the periods. Hence, we obtain the theorem:

The parameters of the points of contact of the four tangents drawn from any point u to the curve are

$$-\frac{u}{2}, -\frac{u+\omega}{2}, -\frac{u+\omega'}{2}, -\frac{u+\omega+\omega'}{2}$$

and conversely, the tangential point (u) of any *point*, (v) is determined by the equation $u+2v\equiv 0$ (mod. ω, ω')

$$i.e., \quad u\equiv -2v \ (\text{mod. } \omega, \omega').$$

If we put $u_1=u_2=u_3$, we obtain the parameters of the nine points of inflexion in the form

$$0, \quad \frac{\omega}{3}, \quad \frac{\omega'}{3}, \quad \frac{\omega+\omega'}{3}, \quad \frac{2\omega}{3}, \quad \frac{2\omega'}{3},$$

$$\frac{2\omega+\omega'}{3}, \quad \frac{\omega+2\omega'}{3}, \quad \frac{2\omega+2\omega'}{3}$$

* A. G. Greenhill, The Applications of the Elliptic Functions, § 144.

These may be expressed symbolically as—

$$(00),\ (10),\ (01),\ (11),\ (20),\ (02),\ (21),\ (12),\ (22)$$

whence we see that the nine points lie on the 12 inflexional lines—

(00) (01) (02)	(00) (10) (20)	(00) (11) (22)	(02) (11) (20)
(10) (11) (12)	(01) (11) (21)	(01) (12) (20)	(00) (12) (21)
(20) (21) (22)	(02) (12) (22)	(02) (10) (21)	(01) (10) (22)

It can be easily seen that the periods

$$\frac{\omega}{2},\ \frac{\omega'}{2} \text{ and } \frac{\omega+\omega'}{2}$$

correspond to the points of contact of the tangents drawn from the inflexion (00) to the cubic, and since the sum of the periods is equal to $\omega+\omega'$, the three points are collinear.

From Abel's theorem it follows that the parameters of the $3n$ points of intersection of a cubic with any *n-ic* satisfy the relation.

$$u_1+u_2+u_3+\dots+u_{3n}\equiv 0 \pmod{\omega,\ \omega'}$$

If now we put $n=2$, we obtain—

$$u_1+u_2+u_3+u_4+u_5+u_6\equiv 0 \pmod{\omega,\ \omega'}$$

i.e., the sum of the arguments of the six points of intersection of a conic with a cubic is congruent to zero.

If now we put $u_1=u_2=u_3=u_4=u_5=u_6$, we obtain

$$u_1=\frac{m\omega+m'\omega'}{6}$$

which gives the parameters of 36 points by putting m and m' equal to 0, 1, 2, 3, 4, 5, and these include the points of inflexion.

Hence, rejecting those nine points, we obtain the parameters of the 27 points where the conic has six-pointic contact with the cubic, *i.e.*, the 27 *sextactic* points (§ 21)

Putting $u_1=u_4$, $u_2=u_5$, $u_3=u_6$, we obtain

$$u_1+u_2+u_3=\frac{m\omega+m'\omega'}{2}$$

which shows that the three points u_1, u_2, u_3 are the points of contact of a conic with the cubic, and hence there are three systems of conics.

It will be seen that the curve is bi-partite, the odd circuit is given by real values of the parameter u lying between 0 and ω, and the even circuit is given by $u=v+\omega'$ where v is real, lying between 0 and ω.

Various other properties of intersections of a cubic with other curves can be established in a similar manner, for which the student is referred to Clebsch, Leçons sur la Geometrie, Vol. II, Chap. II, pp. 355-393.

Ex. 1. Shew that the tangential of a sextactic point is an inflexion.

[If v is the parameter of the tangential,

$$u+u+v\equiv 0 \quad \text{or} \quad v\equiv -2u \ (\text{mod. } \omega, \omega')]$$

Ex. 2. Shew that there are 72 *coincidence* points on a cubic (§ 21).

[The parameter of such a point satisfies $9u\equiv 0(\text{mod. } \omega, \omega')$

$$\therefore \quad u=\frac{m\omega}{9}+\frac{m'\omega}{9} \ (m, m'=0, 1, 2, 3)\ ]$$

Ex. 3. Shew that a point (u) coincides with its third tangential if $9u\equiv 0$.

Ex. 4. Shew that six of the coincidence points are real, all lying on the odd circuit.

Ex. 5. A conic has five-point contact with a cubic Shew that it meets the cubic again on the line joining the point of contact to its second tangential.

Ex. 6. Shew that a cuspidal cubic has no sextactic point, nor coincidence point.

Ex. 7. Prove that the locus of the sextactic points of a pencil of cubics is the nine harmonic polars.

82. Particular cases:

When $k=0$, the cubic (1) of § 80 reduces to the form

$$z^2x=y^2(y-x)$$

which has an acnode at (1, 0, 0). In this case the elliptic functions reduce to ordinary circular functions,* and the co-ordinates are expressed as—

$$\sin^3 u : \sin u : \cos u$$

since, when $k=0$,

$$\mathrm{sn}u=\sin u, \quad \mathrm{cn}u=\cos u, \quad \mathrm{dn}u=1.$$

Similarly, when $k=1$, the elliptic functions reduce to hyperbolic functions and the equation of the cubic becomes

$$z^2x=y(y-x)^2$$

which has a crunode at (1, 1, 0) and the co-ordinates are—

$$\tanh^3 u : \tanh u : \mathrm{sech}^2 u, \qquad \text{since, when } k=1,$$

$$\mathrm{sn}u=\tanh u, \quad \mathrm{cn}u=\mathrm{sech}\, u, \quad \mathrm{dn}u=\mathrm{sech}\, u.$$

Thus if the cubic is unicursal, the co-ordinates are expressible in terms of circular or hyperbolic functions.

83. Use of Weierstrass's $\wp$ function:

We have explained how the expressions of co-ordinates of any point on a curve are simplified by using Weierstrass's $\wp$ function.† If the equation of a non-singular cubic is put into the form—

$$y^2z=4x^3-g_2xz^2-g_3z^3$$

it is at once found that any point on the cubic may be taken as—

$$\wp u, \wp' u, 1.$$

* Greenhill, *loc. cit.* § 16.

† Theory of Plane Curves, Vol. I, § 244.

It follows that three points with parameters u, v, w on the cubic are collinear,

if
$$\begin{vmatrix} \wp u & \wp' u & 1 \\ \wp v & \wp' v & 1 \\ \wp w & \wp' w & 1 \end{vmatrix} = 0$$

whence, as proved in works on elliptic functions,

$$u+v+w \equiv 0 \text{ (mod. } \omega, \omega')$$

where ω, ω' are the periods.

For other particulars with regard to intersections of a non-singular cubic with a conic, etc., see Clebsch—*Sur Theorie der Curven dritter Ordnung*—Crelle, Bd. 63 (1864), pp. 115-121.

Ex. Shew that the four tangents drawn from any point of the cubic form a pencil, having one of the cross-ratios equal to k^2.

84. STEINER'S POLYGON:*

The formulae we have obtained before may be applied to the investigation of Steiner's Polygon inscribed in a unicursal curve.

Let p and q be the parameters of a Steiner's pair.

Through p draw any straight line meeting the cubic in two points u_1 and u_2 (say). Joining u_2 to q, we obtain a

* Application to a crunodal cubic is to be found in a paper by Weyz—Über Kurven dritter Ordnung mit einem mittelpunkt—Math. Ann. Bd. 3 (1871). See also G. Loria, *loc. cit.*, p. 29.

For application of elliptic functions in studying the properties of Steiner's Polygon—see Clebsch, Crelle, Bd. 63 (1864), pp. 106-121.

third point u_3 in which this meets the cubic. Let u_4 be the third point where the line $u_3 p$ meets the cubic, and so on. We thus obtain, by $2n$ operations, $2n+3$ points on the curve

$$(p, q, u_1, u_2, \ldots u_{2n}, u_{2n+1})$$

having one of the following groups of $2n$ conditions satisfied by their parameters, according as the curve has a node,* a cusp or a conjugate point:

$$\left.\begin{array}{ll} pu_1u_2=k & qu_2u_3=k \\ pu_3u_4=k & qu_4u_5=k \\ \cdots\cdots & \cdots\cdots \\ pu_{2n-1}u_{2n}=k, & qu_{2n}u_{2n+1}=k. \end{array}\right\} k=\text{const.} \quad \ldots \quad (1)$$

$$\left.\begin{array}{ll} p+u_1+u_2=0 & q+u_2+u_3=0 \\ p+u_3+u_4=0 & q+u_4+u_5=0 \\ \cdots\cdots & \cdots\cdots \\ p+u_{2n-1}+u_{2n}=0, & q+u_{2n}+u_{2n+1}=0 \end{array}\right\} \quad \ldots \quad (2)$$

$$\left.\begin{array}{ll} p_1+u_1+u_2\equiv 0 & q_2+u_2+u_3\equiv 0 \\ p_1+u_3+u_4\equiv 0 & q_2+u_4+u_5\equiv 0 \\ \cdots\cdots & \cdots\cdots \\ p_1+u_{2n-1}+u_{2n}\equiv 0, & q_2+u_{2n}+u_{2n+1}\equiv 0 \end{array}\right\} (\text{mod}\,\pi) \quad \ldots \quad (3)$$

If these operations lead to a closed polygon, we must have $u_{2n+1}=u_1$. Substituting in the equations (1), (2), (3) and multiplying the n equations in (1), or adding the n vertical conditions (2) or (3), we obtain the following conditions for the Steiner's pair, namely,

$$p^n=q^n \quad \ldots \quad (4)$$

$$p=q \quad \ldots \quad (5)$$

$$np\equiv nq \ (\text{mod}\ \pi) \quad \ldots \quad (6)$$

* See Loria, *loc. cit.*, § 19(*a*), p. 28.

whence, if one of p or q is known, the other can be found. If n is odd, from (4) we get only the real root $p=q$, but if n is even, we have also $p=-q$.

Thus, *in a crunodal cubic curve there are Steiner's polygons the number of whose sides is a multiple of 4; if one such number is given, and any point be taken on the curve, then there is only one point on the curve which together with the given point forms a Steiner's pair.*

The equation (5) is satisfied by $p=q$ only. Hence, *on a cuspidal cubic, there is no Steiner's Polygon in the ordinary sense.*

But from (6) we have—

$$p=q+k\frac{\pi}{n}, \ (k=1, 2, 3, \ldots n-1)$$

Hence, *on an acnodal cubic, there are Steiner's Polygons of even order $2n$; if one point is given, there are $n-1$ other points each of which forms with it the fundamental pair of a Steiner's $2n$-gon.*

Ex. 1. Two points having the same tangential point always generate a Steiner's quadrilateral.

Ex. 2. If P' and Q' are the tangentials of two points P and Q and if PQ' and QP' intersect on the curve, then P, Q generate a Steiner's Hexagon.

85. In the case of a non-singular cubic, we may study Steiner's polygons, as we have done in the case of cubics of deficiency zero. It is easily seen, that there is an infinite number of polygons of $2n$ sides and $2n$ vertices, such that the vertices are situated on a curve of the third order and the odd sides all pass through a point p and the even sides through another point q on the same curve.

When one of p or q is given, the other can be determined by means of equations as given in (3), whence in the present case, we have

$$np+\Sigma u_i\equiv 0, \qquad nq+\Sigma u_i\equiv 0$$

i.e.

$$p=q+\frac{1}{n}(m\omega+m'\omega')$$

i.e., p is obtained by means of division of the elliptic function into n parts. In fact,

if

$$q=\beta/n, \quad p\equiv\frac{\beta+m\omega+m'\omega'}{n}.$$

Hence the co-ordinates of p will be obtained by replacing β by

$$\frac{\beta+m\omega+m'\omega'}{n}$$

in the expressions sn, cn, etc.

The problem has n^2 solutions, n being an odd integer, and these solutions may be represented by radicals by means of roots of an equation of order $(n+1)$. Among the n^2 solutions, there is one when $p=q$, and that will not give us any result.

Hence, *if n is an integer, there are n^2-1 points q forming with a given point p a Steiner's pair.**

* For further information, see Clebsch—Leçons, etc., Vol. II, p. 374, etc.

CHAPTER IV

CIRCULAR CUBICS

86. CIRCULAR CUBIC :

DEFINITION : A circular cubic is a curve of the third order, which passes through the two circular points at infinity.

General properties of circular cubics are best studied in connection with those of bicircular quartics; for, they may be regarded as degenerate forms of the latter. In the present chapter, however, we shall study the properties of certain well-known curves belonging to this class as inverses of conics with respect to a point on the curve. For other informations and fuller details, the student is to consult the paper by C. A. Bjerkness—*Sur une certaine classe de courbes de troisiéme ordre rapporteés à lignes droites, qui dépendent de paramèters donnes*—Crelle, Bd. 55 (1853) and also. F. Fricke—*Über ebene Kurven dritter ordnung welche durch die imaginären Kriespunkte gehen*—Dissertation, Jena (1898).

87. THE EQUATION OF A CIRCULAR CURVE :

Let $C=0$ be a circle and $S=0$ a conic. Then the equation of a circular cub ic —

$$u_1 C + kS = 0 \qquad \dots \quad (1)$$

where $k=0$ is the line at infinity and u_1 is a right line. For, this is a cubic curve and evidently passes through the points where the line at infinity intersects the circle, *i.e.*, the two circular points at infinity.

Again, since the line u_1 meets the cubic at one point at infinity and two points at a finite distance on the conic S, it is parallel to an asymptote of the curve.

The general Cartesian equation of a circular cubic can therefore be written as—

$$(lx+my+n)\ (x^2+y^2+2gx+2fy+c)+S=0 \qquad \dots \quad (2)$$

where $S=0$ is the general equation of a conic ;

or, in the form—

$$(x^2+y^2)(lx+my)+V=0 \qquad \dots \quad (3)$$

where V is a conic.

If the line u_1 is taken as the axis of y and the origin on the curve, the equation takes the form—

$$x\ (x^2+y^2)+ax^2+2hxy+by^2+kx=0 \qquad \dots \quad (4)$$

It can be easily seen that a circular cubic is determined by six or seven points, according as it is of deficiency 0 or unity.

Ex. 1. Shew that the equation of any circular cubic can be put in the form

$$x(x^2+y^2)+ax^2+2hxy+2gx+2fy=0$$

Ex. 2. Shew that the equation of a circular cubic having two imaginary inflexions at the circular points is—

$$u(x^2+y^2)+I^3=0.$$

88. Circular cubics as inverses :

The inverse of a conic with respect to a point on the curve is a circular cubic, whose real asymptote is parallel to the tangent to the conic at the centre of inversion.

The equation of a conic having the origin on the curve is of the form—

$$ax^2+by^2+2hxy+2fy+2gx=0 \qquad \dots \quad (1)$$

The inverse of this with respect to the origin* is—

$$\frac{k^4}{r^4}(ax^2+2hxy+by^2)+\frac{k^2}{r^2}(2fy+2gx)=0;$$

or, $$r^2(2fy+2gx)+k^2(ax^2+2hxy+by^2)=0 \quad \dots \quad (2)$$

which is a circular cubic, and the line

$$2fy+2gx=0$$

which is the tangent to the conic at the origin, is parallel to the real asymptote of the cubic.

It is evident from (2) that the origin is a double point on the cubic, which is a node, a cusp, or a conjugate point, according as the conic is an hyperbola, a parabola, or an ellipse.

Conversely, *the inverse of a circular cubic with regard to the double point is a conic through the double point.*

Ex. 1. Shew that the cubic

$$x(x^2+y^2)+ax^2+2hxy+by^2+kx=0$$

is self-inverse with respect to the circle $x^2+y^2=k$.

Ex. 2. Shew that a circular cubic is self-inverse *w. r. t.* four mutually orthogonal circles.

Ex. 3. A nodal circular cubic is self-inverse *w. r. t.* two circles, and a cuspidal circular cubic *w. r. t.* one circle.

Ex. 4. If a circular cubic is inverted *w. r. t.* any point on the curve, prove that the cross-ratio of tangents is the same for the curve and its inverse.

[The tangents from I invert into the tangents from J forming a pencil with the same cross-ratio.]

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* Theory of Plane Curves, Vol. I, § 15.

89. Special cases:

The equation of a conic having the origin at the vertex may be written as—

$$ax^2+by^2=x$$

the inverse of which is—

$$x(x^2+y^2)=k^2(ax^2+by^2)=ax^2+\beta y^2 \qquad \ldots \quad (1)$$

The following special cases are to be noticed:

(1) When a and b have the same sign, the curve is the inverse of an ellipse, and when they are of opposite signs, it is the inverse of an hyperbola.

(2) When $a=0$, the cubic is the inverse of a parabola and is called a *Cissoid.*

(3) When $a+b=0$, the curve is the inverse of a rectangular hyperbola and is called the *Logocyclic cubic.*

The curve (1) has a double point at the origin, the tangents at which are—

$$ax^2+\beta y^2=0.$$

Therefore, the origin is a node, a cusp, or a conjugate point, according as—

$$ax^2+by^2=0$$

represents two real and distinct, coincident, or imaginary lines; *i.e.*, according as the conic is an hyperbola, a parabola, or an ellipse. The curve again cuts the axis of x at the point $x=a$, which is called the *vertex.* The only real asymptote to the curve is $x=\beta$, and the curve has a real point of inflexion at infinity.

Ex. 1. Shew that a cubic can always be projected into a circular cubic.

Ex. 2. Shew that the tangents to a circular cubic from a point where the tangent is parallel to the real asymptote are all equal.

90. THE LOGOCYCLIC CURVE:

The Cartesian equation of the curve is—

$$x(x^2+y^2)=a(x^2-y^2) \quad \ldots \quad \ldots \quad (1)$$

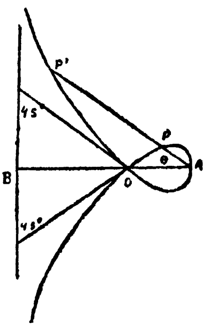
The nodal tangents are—

$$x^2-y^2=0$$

which are two right lines at right angles to each other. The only real asymptote is—

$$x+a=0$$

which, therefore, makes an angle of 45° with each nodal tangent.



The polar equation of the curve, referred to the node, is—

$$r \cos \theta = a \cos 2\theta$$

Transferring the origin to the vertex A, the polar equation reduces to—

$$r^2+2ar \sec \theta + a^2=0$$

or $$r=-a(\sec \theta \pm \tan \theta) \quad \ldots \quad (2)$$

or, changing the constant

$$r=a(\sec \theta \pm \tan \theta). \quad \ldots \quad (3)$$

91. PROJECTION OF A NODAL CUBIC :

THEOREM : *Every nodal cubic can be projected into a logocyclic curve.*

Consider a nodal cubic having the node at O with the nodal tangents OA and OB. Two lines OP and OQ, the harmonic conjugates of OA and OB, meet the curve each in one other point P and Q (say). Let PQ intersect the curve in a third point R, the tangent at which meets the nodal tangents in A and B respectively. Now project the figure, so that the line PQ goes off to infinity, and at the same time, the angles OAB and OBA are each projected into an angle of 45°. Then the prcjected figure will be a *logocyclic** curve.

For, a node is projected into a node and the nodal tangents into nodal tangents. The line AB is projected into the asymptote, which makes with each nodal tangent an angle of 45°. Therefore, the angle between the nodal tangents is a right angle, and since OP and OQ are harmonic conjugates of OA and OB, and also harmonic properties are unaltered by projection, OP and OQ are projected into the two circular lines.† Thus, the projected curve is a circular cubic, the points P and Q being projected into the two circular points at infinity.

92. GEOMETRICAL CONSTRUCTION :

Let O be a fixed point and AB a fixed right line. Draw OA perpendicular on AB, and any right line OP through O intersecting AB at B. On OP take two points P, P′ such that

$$PB = P'B = AB.$$

* Properties of logocyclic cubics have been discussed by Dr. Booth in connection with the geometrical treatment of logarithms, a detailed account of which will be found in his book—Treatise on Some Geometrical Methods.

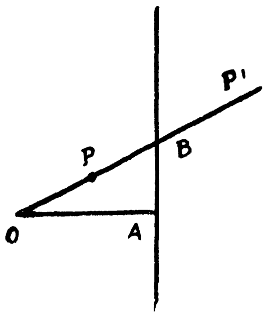
† C. V. Durell, Geometry for Advanced Students, Vol. II, p. 31.

Then the locus of P and P′ is the logocyclic cubic, of which O is the vertex, and A the node.

For, if $\angle AOP=\theta$, we have—

$$OP=OB-PB=OB-AB$$

$$=OA(\sec\theta+\tan\theta)$$



$\therefore$ The polar equation of the locus is—

$$r=a(\sec\theta\pm\tan\theta)$$

which is the equation of the logocyclic cubic, with vertex at the origin and A as the node. (§ 90)

93. Symmetrical cubics:

A cubic curve has, in general, no symmetry like curves of the second order, but it can be projected in such a manner that the projection is always a symmetrical curve.

Thus a nodal cubic can be projected into a Folium of Descartes, which is a curve symmetrical about a line. A cuspidal cubic can be projected into a semi-cubical parabola, and so on.

Theorem: *Any cubic, other than a crunodal or cuspidal cubic, can be projected so as to have the symmetry of an equilateral triangle.**

* Clifford, Coll. Works, p. 412.

Since a non-singular cubic or an acnodal cubic has three collinear real inflexions and three real inflexional tangents, the equation of the curve may be written as—

$$(lx+my+nz)^3+6kxyz=0$$

Now, project the curve so that the real inflexional line goes to infinity and the real triangle formed by the inflexional tangents is equilateral. Then, the equation of the projection is

$$(x+y+z)^3+6kxyz=0,$$

$$x+y+z=0$$

being the line at infinity, and this shows that the curve is symmetrical about the triangle of reference.

Ex 1. Shew how a cubic can be projected so as to become symmetrical about a line or a point.

Ex. 2. Discuss the case when the inflexional tangents are concurrent.

Ex. 3. Any circular cubic can be inverted into a circular cubic symmetrical about a line.

94. THE TRISECTRIX OF MACLAURIN:

The equation of the Trisectrix of Maclaurin is—

$$x(x^3+y^3)=a(y^2-3x^2) \qquad \cdots \quad (1)$$

Therefore, the curve is a circular cubic, the origin being a node. The nodal tangents are $y\pm\sqrt{3}x=0$, each making an angle of 60° with the real asymptote. The circular points are points of inflexion on the curve, For, any circular line is of the form—

$$y=\pm ix+h \qquad \ldots \quad (2)$$

which intersects (1), where

$$x\{x^2+(\pm ix+h)^2\}=a(\pm ix+h)^2-3x^2\}$$

i.e., $$x^2(\pm 2ih+4a)+x(h^2\pm 2iah)-ah^2=0 \qquad \ldots \quad (3)$$

Evidently one intersection is at infinity; a second will be at infinity,

if $\pm 2ih+4a=0$, *i.e.* if $h=\pm 2ia$

But then the co-efficient of x also vanishes. Hence the three points of intersection coincide at infinity, if $h=\pm 2ia$.

Thus the lines $y=\pm ix\pm 2ia$ are inflexional tangents to the curve, the points I and J being two imaginary points of inflexion.

The line at infinity intersects the curve in one other point R, which must be a real point of inflexion. Writing the equation in the form—

$$(x-a)(x^2+y^2)+a(x^2+y^2)-a(y^2-3x^2)=0$$

or,

$$(x-a)(x^2+y^2)+4ax^2=0$$

it is seen that $x=a$ is the tangent at the third inflexion R.

Hence we may define the Trisectrix of Maclaurin as *a nodal circular cubic which has two points of inflexion at the circular points.*

The equation of a circular cubic, having the node at the origin with the real asymptote parallel to the axis of y, becomes—

$$x(x^2+y^2)+ax^2+2hxy+by^2=0 \quad \dots \quad (1)$$

The constants a, h, b, are to be determined from the condition that the circular points will be points of inflexion on the curve, *i.e.*, from the condition that any circular line meets (1) in three coincident points at infinity.

These give $h=0$ and $a+3b=0$.

Therefore, the equation (1) reduces to—

$$x(x^2+y^2)=b(3x^2-y^2).$$

Putting $-a$ for b, we obtain—

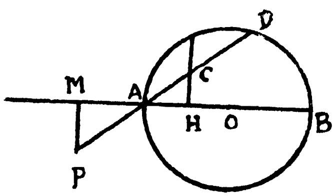
$$x(x^2+y^2)=a(y^2-3x^2)$$

Hence, *every nodal cubic can be projected into a Trisectric of Maclaurin*, by projecting any two points of inflexion into the two circular points at infinity.

95. GEOMETRICAL CONSTRUCTION :

The Trisectrix of Maclaurin may be geometrically constructed by the following method :—

Let O be the centre of a circle and AOB a diameter. Through the middle point H of OA draw HC perpendicular to OA. Through A draw the line ACD cutting HC at C, and the circle at D. On DA produced, take a point P, such that AP=CD. Then the locus of P is a Trisectrix of Maclaurin.



For, let A be the origin, and the co-ordinates of P be (x, y). Also let $\angle \text{BAD}=\theta$ and $\text{AO}=2a$.

Now,

$$\text{CD}=\text{AD}-\text{AC}=2\text{ AB}\cos\theta-\text{AH}\sec\theta=4a\cos\theta-a\sec\theta$$

$$\therefore -x=\text{AP}\cos\theta=\text{CD}\cos\theta$$

$$=a(4\cos^2\theta-1)$$

and

$$-y=\text{AP}\sin\theta=\text{CD}\sin\theta$$

$$=a(4\cos^2\theta-1)\tan\theta$$

whence eliminating θ, we obtain—

$$x(x^2+y^2)=a(y^2-3x^2)$$

which is the Trisectrix of Maclaurin.

96. THE FOLIUM OF DESCARTES:

The equation of the curve is—

$$x^3+y^3=3axy \qquad \dots \quad (1)$$

There is symmetry about the line $y=x$, and the tangents at the origin are the axes of co-ordinates.

The equation of the curve can be written as—

$$x^3+y^3+3xy(x+y)=3xy(x+y+a)$$

or,
$$(x+y)^3=3xy(x+y+a) \qquad \dots \quad (2)$$

which shows that the curve has three points of inflexion lying on the line $x+y=0$, and the inflexional tangents are—

$$x=0, \quad y=0, \quad \text{and} \quad x+y+a=0.$$

But $x=0$ and $y=0$ are the nodal tangents at the origin and cannot therefore be regarded as inflexional tangents.

The real point of inflexion is given by—

$$x+y=0, \quad x+y+a=0,$$

which meet at infinity.

$\therefore$ The line $\qquad x+y+a=0$

is the real asymptote to the curve; in fact, it is an inflexional tangent, the point of contact lying at infinity.

Again, the equation (1) can be written as—

$$x^3+\omega^3y^3+3\omega xy(x+\omega y)=3xy(\omega x+\omega^2y+a)$$

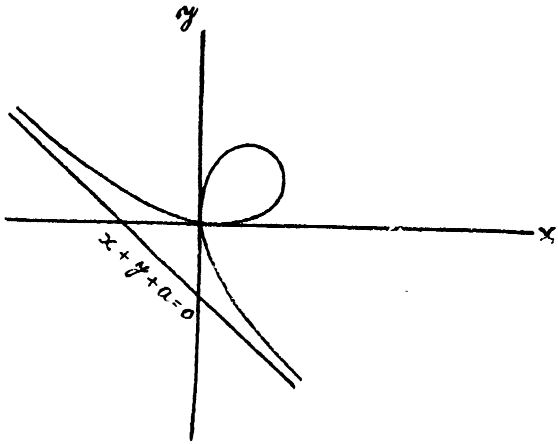
or
$$(x+\omega y)^3=3xy(\omega x+\omega^2y+a)$$

where ω is an imaginary cube root of unity.

This shows, that $\omega x+\omega^2y+a=0$ is an inflexional tangent, the point of contact being at infinity.

Similarly, $\omega^2 x+\omega y+a=0$ is another inflexional tangent, whose point of contact also lies at infinity. Hence, all the three points of inflexion lie at infinity.

The real asymptote is the line $x+y+a=0$, which evidently makes an angle of 45° with each of the axes, *i.e.*, the nodal tangents. The form of the curve is almost similar to that of the logocyclic cubic and is shown in the accompanying figure.



Hence, the Folium of Descartes is a nodal cubic, whose three points of inflexion lie at infinity and the tangent at one of these points makes an angle of 45° with the nodal tangents.

Thus, *every nodal cubic can be projected into the Folium of Descartes.*

Let ABC be the inflexional line and let the tangent at the real inflexion cut the nodal tangents at P and Q, O being the node. Project the figure such that ABC goes off to infinity and the angles OPQ and OQP are projected each into an angle of 45°. Then the nodal tangents of the projected figure are mutually orthogonal and the points of inflexion are at infinity.

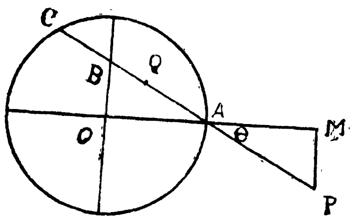
97. GEOMETRICAL CONSTRUCTION.

If the axes be turned through an angle of 45°, the equation of the curve becomes—

$$x(x^2+3y^2)=a(x^2-y^2) \quad \ldots \quad (3)$$

The form (3) suggests the following method of generating the curve geometrically.

Let O be the centre of a circle, OA and OB being two perpendicular diameters. Draw ABC any line intersecting OB at B and the circle at C. On AB take a point Q, such that AQ=BC. Let P be the harmonic conjugate of Q with regard to A and B. Then the locus of P is a Folium of Descartes.



Let OA$=a$, $\angle$ PAM$=\theta$, and P be the point (x, y).

We have,
$$\frac{1}{\text{AP}}+\frac{1}{\text{AB}}=\frac{2}{\text{AQ}} \quad \ldots \quad (4)$$

AB$=a\sec\theta$ and AQ=AC$-$AB$=2a\cos\theta-a\sec\theta$.

whence,
$$\text{AP}=\frac{(2a\cos\theta-a\sec\theta)}{(3-2\cos^2\theta)}$$

$$=\frac{a\cos 2\theta}{\cos\theta\,(3-2\cos^2\theta)}$$

$$\therefore \quad x=\text{AM}=\text{AP}\cos\theta=\frac{a\cos 2\theta}{3-2\cos^2\theta} \quad \ldots \quad (5)$$

and
$$y=\text{PM}=\text{APSin}\,\theta=\frac{a\cos 2\theta}{3-2\cos^2\theta}\tan\theta \quad \ldots \quad (6)$$

Eliminating θ between (5) and (6) we obtain—

$$a(x^2-y^2)=x(x^2+3y^2)$$

which is the Folium of Descartes.

It is to be noticed that the locus of Q is the logocyclic cubic; for, if (x, y) be the co-ordinates of Q,

$$\frac{x^2-y^2}{x^2+y^2}=\cos 2\theta=\frac{\mathrm{AQ}\cos\theta}{a}=\frac{x}{a}$$

$$\therefore \quad x(x^2+y^2)=a(x^2-y^2)$$

which is the logocyclic cubic.

98. THE CISSOID:

The Cissoid is the inverse of a parabola with respect to its vertex.

The inverse of the parabola $y^2=4ax$, with respect to its vertex, is—

$$x(x^2+y^2)=\frac{k^2}{4a}y^2$$

i.e. $\quad x(x^2+y^2)=by^2$, where $b=k^2/4a$.

This curve is also the first positive pedal of the parabola

$$y^2+4bx=0$$

with respect to its vertex. For,

if $\quad x\cos\omega+y\sin\omega=p$

is a tangent to the parabola, we have—

$$p\cos\omega=b\sin^2\omega.$$

$\therefore$ The polar equation of the pedal is $r\cos\theta=b\sin^2\theta$, which gives the Cartesian equation in the form—

$$x(x^2+y^2)=by^2,$$

which is the Cissoid.

The origin is a cusp with $y=0$ as the cuspidal tangent.

The Hessian is $xy^2=0$, which shows that $x=0$ and $y=0$ are inflexional tangents. But the origin being a cusp, it cannot be regarded as a point of inflexion. Hence, the only point of inflexion lies at infinity on the line $x=0$. $x=b$ is consequently an inflexional tangent cutting the cuspidal tangent at a right angle.

Writing the equation in the homogeneous form—

$$4(x-bz)\{y^2+(x+\tfrac{1}{2}bx)^2\}+b^2(3x+bz)z^2=0$$

the real asymptote is found to be $x=b$, and the two imaginary asymptotes are given by—

$$y^2+(x+\tfrac{1}{2}b)^2=0$$

i.e., a pair of circular lines meeting at the point $(-b/2, 0)$.

Thus, the Cissoid is a cuspidal circular cubic, whose point of inflexion is at infinity, and the inflexional tangent is the real asymptote, making a right angle with the cuspidal tangent.

Hence *every cuspidal cubic can be projected into a Cissoid.*

Let OB be the cuspidal tangent at the cusp O and AB the inflexional tangent at A. Let OP and OQ, the harmonic conjugates of OA and OB, meet the curve at P and Q. Then A, P, Q are collinear.

For, the two harmonic pencils O(APBQ) and B(APOQ) have the self-corresponding ray OB.*

Now, projecting P and Q into the two circular points at infinity, A will be projected into the point of inflexion at infinity, and the inflexional tangent will be at right angles to the cuspidal tangent.

* Reye, Geometry of Position, § 86.

Ex. 1. Express the co-ordinates of any point on the Cissoid in terms of a single parameter.

$$[x=b/(1+t^2), \quad y=b/t(1+t^2)]$$

Ex. 2. Shew that the sum of the parameters of four concyclic points P_1, P_2, P_3, P_4 on the Cissoid (cusp at O) is zero.

Prove also the sum of the cotangents of the angles made by OP_1, OP_2, OP_3, OP_4 with the cuspidal tangent is zero.

Ex. 4. The chord PQ of the Cissoid $x(x^2+y^2)=by^2$ subtends a right angle at the cusp. Show that

(*i*) The locus of the mid-point of PQ is a straight line.

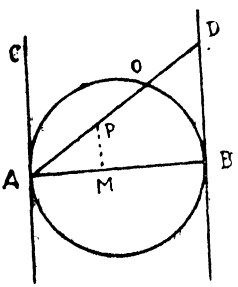
(*ii*) The locus of the intersection of the normals at P and Q is a straight line, while the locus of the intersection of the tangents at those points is a circle.

Ex. 5. If the cusp be joined to three collinear points, the sum of the cotangents of the angles made by these lines with the cuspidal tangent is zero.

99. Geometrical construction:

The Cissoid may be geometrically constructed as follows :—

Let AB be the diameter of a circle AOB with AC, BD as the tangents at A and B respectively.



Let any line AO cut the circle at O, and the tangent BD at D.

If a point P is taken upon AO, such that AP=OD, the locus of P will be a Cissoid.

Let P be the point (x, y) referred to AB and AC as axes, and put AB$=b$ and $\angle$PAM$=\theta$.

Then, $AP=OD=AD-AO=b\sec\theta-b\cos\theta.$

$$\therefore \quad x=AP\cos\theta=b-b\cos^2\theta;$$

And, $y=AP\sin\theta=b\tan\theta-b\sin\theta\cos\theta$

whence, eliminating θ, we obtain—

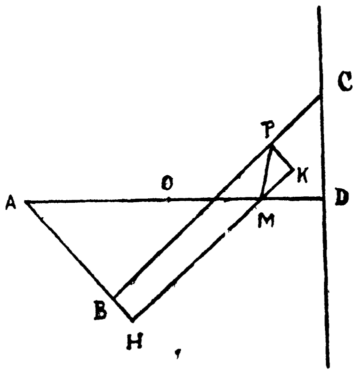
$$x(x^2+y^2)=by^2$$

which is the Cissoid.*

100. NEWTON'S METHOD:

Newton has given the following elegant mechanical construction for this curve by continuous motion:

A right angle has the arm BC of constant length b, the point C moves along a fixed line CD, while the other arm AB passes through a fixed point A at a distance equal to BC from CD.



Then the locus of the middle point P of BC is a Cissoid.†

Take the mid-point O of AD as origin, and $\angle BAD=\theta$.

Then, $x=OM=OD-MD.$

$$\therefore \quad MD=OD-OM=\tfrac{1}{2}b-x.$$

* The Cissoid was invented by the Greek Geometer Diocles in the sixth century, and is better known as the *Cissoid of Diocles*. He made use of this curve in the geometrical construction of two mean proportionals between two given lines, *i.e.*, for solving the famous problem of the *duplication* of the cube.

† Lardner's Algebraic Geometry, pp. 196, 472.

But from the geometry of the configuration, we have—

$$MD = PC \sin\theta = \tfrac{1}{2}b \sin\theta$$

$$\therefore \quad \tfrac{1}{2}b - x = \tfrac{1}{2}b \sin\theta \qquad \ldots \quad (1)$$

Again, $\tfrac{1}{2}b = BP = HK = HM + MK$

$$= (x + \tfrac{1}{2}b)\sin\theta + y\cos\theta \qquad \ldots \quad (2)$$

Eliminating θ between (1) and (2), we obtain—

$$x(x^2 + y^2) = by^2$$

which is the Cissoid.

Various properties of and the different methods of generating the Cissoid have been given by G. Loria in his *Spezielle Alg. und Trans. ebene Kurve*, Vol. I (1910), pp. 36-46, where a number of other references are also to be found.

Ex. 1. O is the vertex and OP a radius vector of a parabola. If a point Q is taken on OP equal to the reciprocal of OP, prove that the locus of Q is a Cissoid.

Ex. 2. A parabola passes through two fixed points P and Q, such that PQ is the normal at P. Shew that the locus of the focus is a Cissoid.

Ex. 3. If a parabola roll on another equal parabola, the locus of the vertex will be a Cissoid.

101. The Cubical Parabola:

The equation of a cubical parabola is $x^3 = a^2y$. The origin is a point of inflexion with $y=0$ as the tangent. Since the equation is satisfied by $x^3=0$ and $a^2=0$, the point at infinity on the line $x=0$ is a cusp, with the line at infinity as the cuspidal tangent, which is the real asymptote.

Hence, *every cuspidal cubic can be projected into a cubical parabola*; and this is done by projecting the cuspidal tangent to infinity.

The tangential equation of the curve is

$$4a^2\xi^3+27\eta=0$$

Hence the curve is its own reciprocal with respect to the point of inflexion.

102. THE SEMI-CUBICAL PARABOLA:

The equation of the semi-cubical parabola is $ay^2=x^3$.

The origin is evidently a cusp, with $y=0$ as the cuspidal tangent. The directions of the asymptotes are given by $x=0$ taken thrice. Therefore the line at infinity is an inflexional tangent, the point at infinity on the curve in the direction $x=0$ being a point of inflexion. Thus the line $x=0$ joining the cusp to the point of inflexion at infinity is perpendicular to the cuspidal tangent.

Thus, a semi-cubical parabola is a cuspidal cubic, having the line at infinity for the inflexional tangent and the cuspidal tangent perpendicular to the line joining the cusp to the point of inflexion.

Hence, *every cuspidal cubic can be projected into a semi-cubical parabola,* by projecting the inflexional tangent to infinity and at the same time projecting the angle between the cuspidal tangent and the line joining the cusp to the point of inflexion into a right angle.

The tangential equation of the curve is $4a\xi^3=27\eta^2$

Hence, the curve is its own reciprocal *w.r.t.* the cusp.

Ex. 1. The semi-cubical parabola is the evolute of a parabola.

Ex. 2. Shew that the reciprocal polar of a semi-cubical parabola *w.r.t.* the focus of the parabola, of which it is the evolute, is the Cissoid.

103. Foci of Circular Cubics :

Foci of circular cubics will be studied in connection with those of bicircular quartics. For the present, we notice that a nodal cubic is of class four, and circular cubics pass through the two circular points at infinity. Therefore, only two tangents can be drawn from each circular point to the curve and consequently the number of real single foci is two. The tangents at the circular points intersect at a singular focus. The tangent at I intersects the two tangents from J in two points, which are also real single (or double imaginary) foci. Similarly, the points where the tangent at J intersects the two tangents from I are real single foci. Thus the real single foci are *four* in number and real singular focus is *one.*

In cuspidal circular cubics, the class is *three* and the cusp replacing the node, the number of real single foci is the same as in a nodal circular cubic. We shall study this more fully in a subsequent chapter.

CHAPTER V

INVARIANTS AND COVARIANTS OF CUBIC CURVES

104. We shall write the general equation in three variables of a cubic in the following form :—

$$U \equiv ax^3 + by^3 + cz^3 + 3fy^2z + 3gz^2x + 3hx^2y$$
$$+ 3iyz^2 + 3jzx^2 + 3kxy^2 + 6mxyz = 0 \quad \ldots \quad (1)$$

We have shown (§ 37) that the equation of all cubics can be reduced to the semi-canonised form—

$$ax^3 + by^3 + cz^3 + 6m'xyz = 0 \quad \ldots \quad (2)$$

and also to the fully canonised form—

$$x^3 + y^3 + z^3 + 6mxyz = 0 \quad \ldots \quad (3)$$

The form (3) shows that the cubic cannot have more than two independent invariants: for if it had one more, it would have two absolute invariants,* which are functions of the co-efficients and consequently functions of m only. Elimination of m between them would lead to a relation involving only the co-efficients in the general equation, which is impossible, since the co-efficients in (1) are all independent.

The two independent invariants of a cubic are of degrees *four* and *six* respectively, and are denoted by S and T. All other invariants can be expressed in terms of these two only.

* E. B. Elliot—Algebra of Quantics, § 29. For a general discussion of the theory and of the different methods of finding the invariants of a cubic, students are referred to Elliot's Algebra of Quantics, Chap. XVI, and to Cayley's *Third Memoir upon Quantics* (1856). See also Aronhold—Crelle, Bd. 55 (1858) and Clebsch-Gordan—Math. Ann. Bd. I & VI.

105. THE INVARIANT S:

It is shown in works on Algebra* that if in the contravariant $\phi(\lambda, \mu, \nu)$ of a quantic U, symbols of differentiation with respect to the variables are substituted for λ, μ, ν respectively, and the new function

$$\phi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

operates upon the given quantic U, the result will be an *invariant* of U (Sylvester).

The Cayleyan is a contravariant of the cubic, and expresses the condition that the line—

$$\lambda x + \mu y + \nu z = 0$$

shall be cut in involution (§ 60) by the system of polar conics

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial y} = 0, \quad \frac{\partial U}{\partial z} = 0.$$

Writing the equation of the Cayleyan in the form †—

$$\phi(\lambda, \mu, \nu) \equiv A\lambda^3 + B\mu^3 + C\nu^3 + 3F\mu^2\nu + 3G\nu^2\lambda + 3H\lambda^2\mu$$
$$+ 3I\nu^2\mu + 3J\nu\lambda^2 + 3K\lambda\mu^2 + 6M\lambda\mu\nu = 0 \quad \ldots \quad (4)$$

and putting

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

for λ, μ, ν respectively, and operating on the general cubic (1), we obtain an invariant S of the fourth degree in the co-efficients.

* Salmon—Higher Algebra, § 139.

† The co-efficients are given in Salmon's H. P. Curves, § 219 and can be calculated by the method of § 388(*a*), Conic Sections.

Thus, $$S \equiv \phi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)U$$

$$\begin{aligned}&=(ag-j^2)f^2-(ac-gj)kf+(cj-g^2)k^2+ch^2f\\ &\quad -hm(3fg+ck)+(hi+2m^2)(jf+gk)\\ &\quad -mi(af+3jk)+i^2ak\\ &\quad -b\{h(cj-g^2)-m(ac-gj)+i(ag-j^2)\}\\ &\quad -(hi-m^2)^2.\end{aligned}$$

The calculation of the co-efficients and the invariant becomes much simplified, if we use the canonical form (3) of the equation.

The Cayleyan of (3) is, as we have obtained in §60,

$$C=m(\lambda^3+\mu^3+\nu^3)+(1-4m^3)\lambda\mu\nu=0.$$

$\therefore$ The invariant S of the canonical form is given by—

$$\begin{aligned}S\equiv&\left[m\left\{\left(\frac{\partial}{\partial}\right)^3+\left(\frac{\partial}{\partial y}\right)^3+\left(\frac{\partial}{\partial z}\right)^3\right\}\right.\\ &\left.+(1-4m^3)\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial z}\right)\right](x^3+y^3+z^3+6mxyz)\\ =&18m+(1-4m^3)6m\\ =&24m(1-m^3)\end{aligned}$$

Thus, the value of S for the canonical form, removing a numerical factor is $m(1-m^3)$; and this is equal to the invariant S of the untransformed cubic (1), multiplied by the fourth power of the modulus of transformation.

Hence it follows that if $S=0$, we have $m=0$, and the equation of the cubic reduces to—

$$x^3+y^3+z^3=0$$

Thus, *the condition that the equation of a cubic may be expressed as the sum of three cubes, and consequently its Hessian may reduce to three right lines is that the invariant S vanishes.**

* Aronhold—Crelle, Bd. 39 (1850), p. 153.

106. THE INVARIANT T:

It is proved in works on Algebra that, if

$$U \equiv ax^3+by^3+\ldots \text{and} \quad V \equiv a'x^3+b'y^3+c'z^3+\ldots$$

are any two quantics of the same order, and I is any invariant of U, then

$$\left(a'\frac{\partial}{\partial a}+b'\frac{\partial}{\partial b}+\ldots \right) \mathrm{I}$$

is an invariant of U and V,* which is again an invariant of U, if V is a covariant of U.

Now, the Hessian of a cubic is a cubic-covariant. Hence, any invariant of the cubic and its Hessian is an invariant of the cubic itself.

If the Hessian of the general cubic (1) be written in the form—

$$a'x^3+b'y^3+c'z^3+3f'y^2z+3g'z^2x+3h'x^2y+3i'yz^2$$

$$+3j'zx^2+3k'xy^2+6m'xyz=0 \qquad \ldots \quad (5)$$

Then, $$\left(a'\frac{\partial}{\partial a}+b'\frac{\partial}{\partial b}+c'\frac{\partial}{\partial c}+\ldots \right) \mathrm{S}$$

is an invariant of the cubic.

The calculation becomes much simplified for the canonical form of the cubic.

The Hessian of the semi-canonised cubic—

$$a'x^3+b'y^3+c'z^3+6m'xyz=0$$

is $$-m'^2(a'x^3+b'y^3+c'z^3)+(a'b'c'+2m'^3)xyz=0,$$

and the invariant is

$$\mathrm{S}=m'(a'b'c'-m'^3)$$

* Elliot—*loc. cit.*, § 19.

Hence, operating with

$$-m'^2\left(a'\frac{\partial}{\partial a'}+b'\frac{\partial}{\partial b'}+c'\frac{\partial}{\partial c'}\right)$$

$$+\tfrac{1}{6}(a'b'c'+2m'^3)\frac{\partial}{\partial m'}$$

on S, and multiplying the result by 6, we obtain—

$$T=(a'b'c')^2-20m'^3(a'b'c')-8m'^6.$$

Hence, by putting $a'=b'=c'=1$ and $m'=m$, we obtain, for the canonical form (3),—

$$T\equiv 1-20m^3-8m^6.$$

This is then the invariant T * of the original cubic multiplied by the sixth power of the modulus of transformation. Since the Hessian belongs to the syzygetic family of cubics, the second Hessian also belongs to the same family and its expression is found to be—

$$4S^2U+TH=0.$$

Hence, when $T=0$, the cubic becomes *harmonic* and the second Hessian coincides with the original cubic.

It will be seen that the anharmonic ratio of the cubic is connected with the invariants S and T (or the absolute invariant S^3/T^2) by the equation

$$\frac{S^3}{T^2}=24\frac{(1-\sigma+\sigma^2)^3}{(1+\sigma)^2(2-\sigma)^2(1-2\sigma)^2}\quad \text{(see § 53.)}$$

* The actual expression is given in Salmon's H. P. Curves, § 221.

107. THE DISCRIMINANT:

Since a cubic has only *two* independent invariants S and T, all other invariants can be expressed as rational and integral functions of these two.*

Thus the discriminant of the ternary cubic is—

$$\Delta \equiv T^2 + 64S^3.$$

Suppose the cubic has a double point at the origin or at the third vertex C. Then the co-efficients of z^3 and z^2 in equation (1) must be zero. Therefore, $c = g = i = 0$, and the invariant S reduces to—

$$2m^2fj - m^4 - f^2j^2,$$

i.e., to $$-(fj - m^2)^2.$$

Also, T becomes—

$$8(f^3j^3 - 3m^2f^2j^2 + 3m^4fj - m^6)$$

i.e., $$T = 8(fj - m^2)^3$$

$$\therefore \quad T^2 + 64S^3 \equiv \Delta = 0.$$

Thus, when the cubic has a double point, the discriminant—

$$T^2 + 64S^3 = 0.$$

For the canonical form, we have—

$$S = m - m^4 \quad \text{and} \quad T = 1 - 20m^3 - 8m^6.$$

$$\therefore \quad T^2 + 64S^3 = (1 - 20m^3 - 8m^6)^2 + 64(m - m^4)^3$$

$$= (1 + 8m^3)^3,$$

which is the discriminant.

For the nodal cubic—

$$x^3 + y^3 + 6mxyz = 0,$$

we get $$S = -m^4 \quad \text{and} \quad T = -8m^6$$

$\therefore \quad T^2 + 64S^3$ is identically zero, as it should be.

* Elliot—*loc. cit.*, § 295.

If the curve has a cusp with $y=0$ as the tangent, we must have further $m=0$ and $j=0$, which give $S=0$, $T=0$. Hence the condition, both necessary and sufficient, that the cubic has a cusp, is that both S and T vanish.

Ex. 1. Shew that the cross-ratio of the pencil of four tangents is given by the equation

$$T^2(\sigma^2-\sigma+1)^2+16S^3(\sigma+1)^2(\sigma+2)^2(2\sigma-1)^2=0. \quad (cf. \S\ 53).$$

Ex. 2. Shew that the invariant S vanishes for all equianharmonic cubics and T vanishes for all harmonic cubics.

Ex. 3. The Hessian of all equianharmonic cubics reduces to three right lines.

Ex. 4. Shew that if $S=0$, the Cayleyan reduces to the double points of the Hessian.

Ex. 5. Find the invariant S of the cubic

$$ax(y^2-z^2)+by(z^2-x^2)+cz(x^2-y^2)=0$$

$$[b^2c^2+c^2a^2+a^2b^2-a^4-b^4-c^4]$$

108. The Sextic Covariant of Cubics :

Since the equation of a cubic contains ten co-efficients and three variables, altogether making up thirteen, and the general scheme of linear transformation contains nine constants, it follows that the number of independent covariants and invariants must be *four*. We have already obtained the two invariants S and T, and one covariant—the Hessian. Therefore, there must be one other *independent* covariant of the cubic.

Again, a covariant of a cubic must be of degree 3, or a multiple thereof, and also a covariant of the canonical form is a linear function of $x^3+y^3+z^3$ and xyz, and consequently of f and H. Therefore, the cubic cannot have any other covariant of degree 3 besides the Hessian. Hence the next covariant must be of degree 6.

Prof. Salmon has given the following sextic covariant * of a cubic :—

If $$ax^2+by^2+cz^2+\ldots=0$$

and $$a'x^2+b'y^2+c'z^2+\ldots=0$$

be the polar conics of a point (x', y', z') with regard to the cubic and its Hessian respectively, then the **F**-conic † of these two is—

$$(BC'+B'C-2FF')x^2-\ldots=0$$

and is a covariant of the two conics. If it passes through the point (x', y', z'), the locus of (x', y', z') is a covariant of the cubic, which is of degree six in the variables and of eight in the co-efficients.

Thus, *the locus of a point such that the F-conic of its polar conics with respect to the cubic and its Hessian passes through the point is a sextic covariant of the cubic.*

The actual expression of this covariant for the general equation has not been calculated, but the expression assumes a simple form for the canonical equation :

The polar conic of a point (x', y', z') with regard to the cubic is—

$$x'(x^2+2myz)+y'(y^2+2mzx)+z'(z^2+2mxy)=0 \quad \ldots \quad (1)$$

and that with regard to the Hessian is—

$$3m^2(x'x^2+y'y^2+z'z^2)$$
$$-(1+2m^3)(x'yz+y'zx+z'xy)=0 \quad \ldots \quad (2)$$

* Salmon—H. P. Curves, § 231.

† Salmon—Conic Sections, § 378.

From the condition that the F-conic of (1) and (2) passes through the point (x', y', z'), we obtain the locus of (x', y', z') as a sextic curve Θ, which is the second fundamental covariant of the cubic given by—

$$\Theta = 3m^2(1+2m^3)(x^3+y^3+z^3)^2$$

$$-m(1-20m^3-8m^6)(x^3+y^3+z^3)xyz$$

$$-3m^2(1-20m^3-8m^6)x^2y^2z^2$$

$$-(1+8m^3)^2(y^3z^3+z^3x^3+x^3y^3)$$

109. Other Covariants:

There are two other covariants of the sixth order of a cubic, any one of which could, with equal justice, be selected as the fundamental sextic covariant. In fact, any one of these being selected as the fundamental covariant, the other two are expressible in terms of this. Dr. Salmon has taken Θ as the fundamental covariant, while Prof. Elliot takes

$$\phi \equiv -(\Theta + 3\text{USH})$$

as the fundamental one.

The other two sextic covariants may be defined as follow:—

(1) *The locus of a point, whose polar line with regard to the Hessian touches the polar conic of the same point with regard to the cubic, is a covariant of the cubic.*

This covariant is found to be*—

$$-4(\Theta + 3\text{SUH}),$$

* *Cf.* Salmon's Conics, § 381, *Ex.* 1

(2) *The locus of a point whose polar line with respect to the cubic touches the polar conic of the same point with respect to the Hessian is a sextic covariant of the cubic.*

This, is found to be

$$-(TU^2-12SUH+4\Theta)$$

Thus we see that there are *three* fundamental covariants of a cubic—namely, H, Θ, and the cubic U itself. All other covariants are expressible in terms of these three.

There is another irreducible covariant of a cubic which was obtained by Brioschi. This covariant for the semi-canonised form is—

$$(abc+8m^3)^3(by^3-cz^3)(cz^3-ax^3)(ax^3-by^3).$$

110. Contravariants of Cubics: *

All contravariants of a cubic can be expressed in terms of three fundamental contravariants. These are the *Evectants* of S and T, and the reciprocal of the cubic, the two evectants being denoted by P and Q respectively, and the reciprocal by F.

Now, the first evectant of S for the semi-canonical form is—

$$P'=\left(\lambda^3\frac{\partial}{\partial a}+\mu^3\frac{\partial}{\partial b}+\nu^3\frac{\partial}{\partial c}+\lambda\mu\nu\frac{\partial}{\partial m}\right)S.$$

Thus,

$$P=m(bc\lambda^3+ca\mu^3+ab\nu^3)+(abc-4m^3)\lambda\mu\nu.$$

$\therefore$ For the fully canonised form this contravariant becomes—

$$P=m(\lambda^3+\mu^3+\nu^3)+(1-4m^3)\lambda\mu\nu=0 \qquad \dots \quad (1)$$

* Cayley, Phil Trans., Vol. 147 (1857), p. 415.

which is the Cayleyan (*Pippian*) of the cubic, as we have already shown.

The evectant of T for the semi-canonised form is—

$$Q'=\left(\lambda^3\frac{\partial}{\partial a}+\mu^3\frac{\partial}{\partial b}+\nu^3\frac{\partial}{\partial c}+\lambda\mu\nu\frac{\partial}{\partial m}\right)T$$

$$=(abc-10m^3)(bc\lambda^3+ca\mu^3+ab\nu^3)$$

$$-m^2(30abc+24m^3)\lambda\mu\nu$$

which, for the canonical form, becomes *—

$$Q=(1-10m^3)(\lambda^3+\mu^3+\nu^3)-m^3(30+24m^3)\lambda\mu\nu \quad \ldots \quad (2)$$

The third contravariant F is the reciprocal of the given cubic. Geometrically its vanishing is the condition that the line

$$\lambda x+\mu y+\nu z=0$$

touches the curve, *i.e.*, $F=0$ is the tangential equation of the cubic.

For the semi-canonical form it is given by—

$$F'\equiv b^2c^2\lambda^6+c^2a^2\mu^6+a^2b^2\nu^6$$

$$-(2abc+32m^3)(a\mu^3\nu^3+b\nu^3\lambda^3+c\lambda^3\mu^3)$$

$$-24m^2\lambda\mu\nu(bc\lambda^3+ca\mu^3+ab\nu^3)$$

$$-24m(abc+2m^3)\lambda^2\mu^2\nu^2\ ;$$

* This is called by Prof. Cayley the *Quippian* of the cubic. No satisfactory geometrical interpretation has been given to this contravariant. One, however, has been given by Cayley—in his "A Memoir on the Curves of the Third Order"—Coll. Papers, Vol. II, p. 396. For a detailed discussion of the theory of invariants and covariants of ternary forms and their applications, the student is referred to Aronhold's Memoirs in Crelle Bd. 39 (1850), p. 140, and Bd. 55 (1858), p. 97, and to Clebsch, Math. Ann., Bd. 1 (1869), p. 56.

and for the canonical form by

$$F \equiv \lambda^6 + \mu^6 + \nu^6 - (2 + 32m^3)(\mu^3\nu^3 + \nu^3\lambda^3 + \lambda^3\mu^3)$$

$$-24m^2\lambda\mu\nu(\lambda^3 + \mu^3 + \nu^3) - 24m(1 + 2m^3)\lambda^2\mu^2\nu^2 \quad \ldots \quad (3)$$

There is another irreducible contravariant, which is not a rational and integral function of these three. This was discovered by Hermite. For the semi-canonical form it is—

$$(abc + 8m^3)^3(c\mu^3 - b\nu^3)(a\nu^3 - c\lambda^3)(b\lambda^3 - a\mu^3);$$

and for the canonical form it becomes—

$$(1 + 8m^3)^3(\mu^3 - \nu^3)(\nu^3 - \lambda^3)(\lambda^3 - \mu^3)$$

CHAPTER VI

Quartic Curves—General Properties

111. The General Equation of Quartic Curves:

The most general equation of a curve of the fourth order in two variables can be written in the form—

$$ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4+fx^3+3gx^2y$$

$$+3hxy^2+iy^3+lx^2+2mxy+ny^2+px+qy+r=0 \quad \ldots \quad (1)$$

or in the symbolic form—

$$u_4+u_3+u_2+u_1+u_0=0 \quad \ldots \quad (2)$$

The general equation therefore contains 14 disposable constants, so that the curve can be made to pass through 14 points chosen arbitrarily, or in other words, 14 points, in general, determine a curve of the fourth order uniquely.

112. Homogeneous Forms:

The equation of a quartic in three variables passing through the vertex A of the fundamental triangle can be put into the form—

$$x^3u_1+x^2u_2+xu_3+u_4=0 \quad \ldots \quad (3)$$

where $u_1=0$ is the tangent at A.

If A is a double point on the curve, the equation takes the form—

$$x^2u_2+xu_3+u_4=0 \quad \ldots \quad (4)$$

where $u_2=0$ represents the two tangents at the double point.

If A is a point of inflexion on the curve, u_2 must contain u_1 as a factor, and the equation of the quartic with A as a point of inflexion becomes—

$$u_1x^3+u_1v_1x^2+u_3x+u_4=0 \qquad \dots \quad (5)$$

Similarly, the equation of a quartic passing through the three vertices of the fundamental triangle assumes the form—

$$x^3(ly+mz)+y^3(l'x+m'z)+z^3(l''x+m''y)$$
$$+ay^2z^2+bz^2x^2+cx^2y^2+2xyz(fx+gy+hz)=0 \quad \dots \quad (6)$$

If the vertices are double points on the curve, the co-efficients of x^3, y^3, z^3 should be absent from the equation, and consequently the equation of a trinodal quartic with nodes at A, B, C can be written as—

$$ay^2z^2+bz^2x^2+cx^2y^2+2xyz(fx+gy+hz)=0$$

i.e., $$\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}+\frac{2f}{yz}+\frac{2g}{zx}+\frac{2h}{xy}=0 \quad \dots \quad (7)$$

113. Classification of Quartic Curves:

The most important question which has been solved in different ways by different geometers is the classification of quartic curves. During the first half of the eighteenth century, this was regarded as a very complicated problem by many workers on geometry. In two long papers * Abbot Bragelogne gave a method, based on the fact that the Cartesian equation of a quartic can be put into canonical forms, but his work does not throw much light upon the subject. After Bragelogne, different workers have given different classifications based on different principles. In fact, there are so many important factors to be taken into account, that an exhaustive list seemed impracticable.

* Examen des lignes du quartiéme ordre ou courbes du troisiéme genre—*Mém. Acad. Sciences*, Paris (1730). Also Note sur les lignes du quartíeme ordre (*ibid*).

114. Classification based on the nature of the points at infinity.

One method of classifying quartic curves, based on the nature of the curve at infinity and its behaviour with the line at infinity, was, however, given by Euler,* Cramer,† and Plücker.‡ Plücker thus gave 152 forms and for each he gave one canonical form of equation.

Accordingly, quartic curves were divided into the *nine* following classes, according to the nature of the four points at infinity on the curve:—

(1) All four imaginary in conjugate pairs; (2) Two real and distinct, and two conjugate imaginaries; (3) Four real and distinct; (4) Two conjugate imaginary and two real and coincident; (5) Two real and distinct, and two real and coincident; (6) Two real double points; (7) Two conjugate imaginary double points; (8) One single and one triple point; (9) A quadruple point.

Each of these classes has again to be subdivided into many other forms, according as the line at infinity, when meeting the curve in two coincident points, is simply a tangent or a line passing through a double point, and this double point may again be of various kinds. Again, the line at infinity may be an ordinary stationary tangent or a tangent at a point of undulation, or may pass through a triple point; or any ordinary intersection may be a point of inflexion or undulation on the curve, in each of these cases, there will be a difference in the figure. Existence at infinity of these higher singularities possible for a quartic curve must be considered in a complete classification of quartics.

* Euler, Introd. in analysin infinitorum II, Lausannae 1748.

† Cramer, Introd. à lànalyse des lignes courbes, Geneve (1750).

‡ Plücker Theorie der algebraischen Kurven, Bonn, 1839.

115. Zeuthen's Classification.*

Zeuthen takes as the basis of his classification of quartic curves the real bitangents of the curve and gives a complete list of the possible forms of non-singular quartics. He begins with the remark that the branches of a curve are of two kinds—odd and even, and a quartic can have only even circuits (ovals), and since the number of circuits can exceed the deficiency only by 1, a quartic can have 1, 2, 3 or 4 ovals. Zeuthen thus divides the bitangents into two classes—(1) the *first* kind, if its points of contact are conjugate imaginary or lie on the same real branch of the curve; (2) *second* kind, if it touches two different branches. There are always four bitangents of the first kind, but the number of those of the second kind is 0, 4, 12 or 24. Hence follows that there are 8 real points of inflexion. Zeuthen calls the branch having no real bitangent or inflexion an *oval*, a branch with 1, 2, 3, or 4 real bitangents,—a *unifolium*, a *bifolium*, a *trifolium* or a *quadrifolium*.

Zeuthen first shows that a conic can be described through the points of contact of four bitangents and puts the equation of a quartic into the form $xyzu=V^2$, where x, y, z, u are the bitangents and V is the conic through their points of contact. He analyses the possible forms of quartics by discussing the different positions of intersections of the four lines with the conic with respect to the quadrilateral formed by them. Thus, when the conic V meets all the lines in real points, Zeuthen divides the quartics into *nine* groups and *thirty-six* species in all, all other possible cases being easily deducible from these.

* H. G. Zeuthen—"Sur les différentes formes des courbes planes du quatrième ordre,—Math. Ann., Bd. 7, p. 411. (1874).

Klein, investigated the real branches of a curve by considering the Riemann surface belonging to the curve and the corresponding Abel's Integral. See Math. Ann., Bd. 10(1876) and Bd. 11(1876).

116. Classification based on deficiency:

One of the most important classifications of quartics, however, is based on deficiency; and they can, therefore, be grouped in four broad divisions, according as the deficiency is 3, 2, 1 or 0. The first group contains no singular points, the second contains one double point (a node or a cusp), the third contains two double points and the last has three double points. The following table shows the *ten* different species, with their corresponding Plücker's numbers, into which the quartic curves can be divided:—

Def. $p=$	Class $m=$	Nodes $\delta=$	Cusps $\kappa=$	Bitang. $\tau=$	Pts. of inflexion $i=$	Degree $n=4$	
3	12	0	0	28	24	4	I
2	10	1	0	16	18	4	II
2	9	0	1	10	16	4	III
1	8	2	0	8	12	4	IV
1	7	1	1	4	10	4	V
1	6	0	2	1	8	4	VI
0	6	3	0	4	6	4	VII
0	5	2	1	2	4	4	VIII
0	4	1	2	1	2	4	IX
0	3	0	3	1	0	4	X

From the above table it is seen that in each of the last four cases the curve is unicursal. In species IX, the curve is also of the fourth class. In this case, therefore, properties of one quartic can be obtained from another by reciprocation. In the last case, a tri-cuspidal quartic is of the third class and therefore its properties can be deduced from those of a nodal cubic by reciprocation.

117. Generation of Quartic Curves:

Just as a curve of order $r+s$ may be generated by means of two projective pencils of curves of orders r and s, a quartic curve may be generated by means of two projective pencils of conics or a pencil of cubics and a pencil of lines projectively related to it.

A second mode of generation of a quartic by means of a conic and a cubic has also been given.* In this method a quartic is the geometrical locus of the poles of the tangents of the conic with respect to the cubic. A third method is based on the theory of bitangents, which we shall discuss in a subsequent section. Construction of a quartic through 14 given points has been given by a number of workers.†

118. Generation by two projective pencils of conics:

Let the two pencils of conics be defined by—

$$P+\lambda Q=0 \qquad \dots \quad (1)$$

and

$$P'+\mu Q'=0 \qquad \dots \quad (2)$$

respectively.

where λ and μ are parameters; and P, Q; P', Q' are the pairs of base-conics of the two pencils.

If the two pencils are homographic, *i.e.*, if there is a (1, 1) correspondence between the members of the pencils, λ and μ must be connected by the homographic relation of the form

$$A\lambda\mu+B\lambda+C\mu+D=0 \qquad (3)$$

* Gerbaldi—*Rend. Circ. Mat.*, Vol. 7, p. 178 (1893).

† Grassmann—*Crelle*, Bd. 44, p 1 (1852), Chasles—*Comp. rend.*, Vol. 37, p. 372 (1853), Jonquières—*Journal de Math.*, Vol. 1 (2), p. 411 (1856).

If, therefore, we eliminate λ and μ between (1), (2) and (3), we obtain—

$$A.\frac{PP'}{QQ'}-B.\frac{P}{Q}-C.\frac{P'}{Q'}+D=0,$$

i.e.,
$$APP'-BPQ'-CP'Q+DQQ'=0 \qquad \dots \quad (4)$$

which, therefore, represents the locus of intersections of the corresponding members of two homographic pencils. The equation (4) is of the fourth degree, and therefore represents a quartic curve.

From the form of the equation it follows that this curve passes through the four base-points of intersection of P and Q, as also through the four base-points of intersection of P′ and Q′. Thus the above quartic curve passes through the eight given fixed points.

Again, the anharmonic ratio of a pencil of lines, which join four fixed points on a conic to a variable point, is constant and is a function of the mutual distances of the four points and the constants in the equation of the conic.* It follows, therefore, that in a pencil of conics $P+\lambda Q$, passing through four fixed points, each individual member is characterised by a definite value of this anharmonic ratio σ, which is a linear function of the parameter λ. For any particular member we have, therefore, $\sigma=k\lambda$, where the value of k depends upon the position of the four base-points. Similarly, for the conics of the second pencil $P'+\mu Q'$, we have $\sigma'=k'\mu$, where the value of k' depends upon the four fixed base-points.

Since the two pencils are homographic, from the relation

$$A\lambda\mu+B\lambda+C\mu+D=0$$

we obtain
$$A.\frac{\sigma\sigma'}{kk'}+B\frac{\sigma}{k}+C\frac{\sigma'}{k'}+D=0,$$

i.e.,
$$A\sigma\sigma'+Bk'\sigma+Ck\sigma'+Dkk'=0.$$

* Salmon—Conic Sections, § 259.

Hence, since k and k' depend upon the base-points and are consequently constants, the above relation may be written as

$$a\sigma\sigma'+b\sigma+c\sigma'+d=0 \quad \ldots \quad \ldots \quad (5)$$

Thus, if T is a point of intersection of two corresponding members of the two pencils, whose base-points are P_1, Q_1, R_1, S_1 and P_2, Q_2, R_2, S_2 respectively, then the anharmonic ratios of the two pencils $T(P_1\ Q_1\ R_1\ S_1)$ and $T(P_2\ Q_2\ R_2\ S_2)$ are connected by the relation (5), and the locus of T is a quartic curve through these base-points.

Therefore, we may define a quartic curve as follows :—

The locus of a point T, such that the cross-ratios of the pencils $T(P_1\ Q_1\ R_1\ S_1)$ and $T(P_2\ Q_2\ R_2\ S_2)$ are connected by a homographic relation, is a quartic curve through the eight fixed points.

Now, the points T, P_1, Q_1, R_1, S_1 lie on the quartic. Therefore, the conic through these five points intersect the quartic in three other points U, V, W, which are also points on the corresponding member of the second pencil. The three points U, V, W possess properties analogous to those of T. Thus the two corresponding conics through P_1, Q_1, R_1, S_1 and P_2, Q_2, R_2, S_2 meet the quartic in four common points T, U, V, W. Hence, the latter group of points is residual to each of the given base-groups. Therefore, the two groups of base-points are co-residual groups on the quartic.

119. From what has been said, it follows that the two groups of base-points are not independent, but are co-residual groups. Hence, in constructing the quartic we proceed as follows :—

Take P_1, Q_1, R_1, S_1 any four arbitrary points on the quartic. Through these four points describe a conic, intersecting the quartic in four other points T, U, V, W. Through these latter points and an assumed point P_2, describe another conic, which will intersect the quartic in

three other points Q_2, R_2, S_2. Thus the group $P_2\,Q_2\,R_2\,S_2$ is co-residual to $P_1\,Q_1\,R_1\,S_1$, and they may be taken as the base-points.

Next consider the case when the base-point P_1 coincides with P_2. Then, two corresponding members of the pencils always intersect at the point $P_1(P_2)$, and consequently in three other points only. The tangents at P_1 (P_2) to the corresponding conics form two homographic pencils of lines, and therefore they have two self-corresponding rays,* Thus there are two pairs of corresponding conics which have a common tangent at $P_1(P_2)$. Now, since the intersections of two corresponding conics lie on the quartic, therefore the common tangent to a pair of corresponding conics is also a tangent to the quartic at P_1 (P_2). Hence the quartic has two tangents at the point P_1 (P_2) which, therefore, is a double point on the curve. Similarly, it can be shown that if Q_1 and Q_2 coincide, Q_1 (Q_2) is a double point on the curve. If further R_1 (R_2) coincide, it is also a double point.

Hence a trinodal quartic may be defined as follows.—

The locus of intersection of two homographic pencils of conics, which have three fixed base-points common, is a quartic having double points at the three common points.

120. COMPLEX SINGULARITIES:

We have seen that quartics can be divided into ten species, according to the nature and number of double points they possess. But there are other special forms, which possess complex singularities, arising from the union of two or more of the ordinary singularities.

(1) A TACNODE: Two nodes may coincide as consecutive points on a curve, giving rise to the singularity called a *tacnode*. The tangent at a tacnode has a contact of the third order with the curve and therefore cannot meet the

* Scott—Modern Analytical Geometry, § 164.

curve in any other point. In fact, two branches of the curve have simple contact at a tacnode. The tacnodal tangent counts as two bi-tangents.

The general equation of a quartic having a tacnode at the origin, with the axis of x as the tacnodal tangent, can be written as

$$y^2+bx^2y+cxy^2+dy^3+ex^4+fx^3y$$
$$+gx^2y^2+hxy^3+iy^4=0 \quad \ldots \quad (1)$$

For, the shape of the curve near the origin is given by—

$$y^2+bx^2y+ex^4=0$$

or,

$$y=x^2\left(\frac{-b\pm\sqrt{b^2-4e}}{2}\right)$$

i.e., $y=\lambda x^2$, and $y=\mu x^2$, where λ and μ are the roots of

$$\rho^2+b\rho+e=0 \quad \ldots \quad (2)$$

Hence, there are two branches of the curve whose forms near the origin are the same as those of the parabolas $y=\lambda x^2$, and $y=\mu x^2$, and consequently the two branches touch each other at the origin, which is therefore a tacnode, with $y=0$ as the tacnodal tangent.

(2) A Rhamphoid cusp: This is formed by the coincidence of an ordinary cusp and a node. This has been called so, from a fancied resemblance to the form of a *beak*. The tangent at such a point counts once as a double tangent and once as a stationary tangent.

The general equation of a quartic having a rhamphoid cusp at the origin is—

$$(y-\lambda x^2)^2+cxy^2+dy^3+fx^3y+gx^2y^2+hxy^3+iy^4=0 \ldots \quad (3)$$

This is a particular case of (1), when the roots of the equation (2) are equal, *i.e.*, when $\lambda=\mu$.

The shape of the curve near the origin is given by—

$$y=\lambda x^2 \pm kx^{\frac{5}{2}} \quad \text{or} \quad (y-\lambda x^2)^2-k^2x^5=0.$$

(3) AN OSCNODE : This is formed by the union of *three* nodes as consecutive points on the curve, and is called an *oscnode*. At an oscnode two branches of the curve have a three-pointic contact.

The general equation of a curve having an oscnode at the origin is—

$$(y-\lambda x^2)^2+cxy(y-\lambda x^2)+dy^3$$
$$+gx^2y^2+hxy^3+iy^4=0 \qquad \ldots \quad (4)$$

The forms of the two branches of the curve near the origin are given by—

$$y=\lambda x^2+k_1x^3 \quad \text{and} \quad y=\lambda x^2+k_2x^3,$$

where k_1 and k_2 are the roots of—

$$k^2+c\lambda k+\lambda^2(d\lambda+g)=0.$$

These show that the two branches cross as well as touch each other at the origin. The oscnodal tangent counts as three bi-tangents, as there is only one other ordinary bi-tangent.

(4) A TACNODE-CUSP : It is formed by the union of two nodes and a cusp, or a tacnode and a cusp, as consecutive points on a curve. The general equation of a curve having a tacnode-cusp at the origin is of the form—

$$(y-\lambda x^2-cxy-dy^2)^2=\mathrm{A}xy^3+\mathrm{B}y^4.$$

(5) A TRIPLE POINT : Three double points may coincide at a point giving rise to a *triple point*. The general equation of a quartic having a triple point at the origin is of the form $u_3+u_4=0$, the three tangents at the origin being given by $u_3=0$. This is a cubic in y/x. Four distinct cases are to be considered according as the tangents are

(*i*) all real and distinct, (*ii*) one real and distinct and two real and coincident, (*iii*) all real and coincident, (*iv*) one real and two imaginary. Hence there are four *species* of triple points.

(6) A FLECNODE AND A BIFLECNODE: A node may be a point of inflexion on one or both branches of the curve passing through it. The two cases are distinguished as follow:—

(*a*) A FLECNODE: It is a node, which is a point of inflexion on one branch of the curve, and consequently, one of the tangents at which is a stationary tangent. Such a point may be considered as arising from the union of an ordinary node with a point of inflexion. Every flecnodal tangent has a four-pointic contact with the curve, since it has a three-pointic contact with the branch it touches and cuts the other branch. The equation of a quartic having a flecnode at the origin is—

$$u_1v_1+u_1v_2+u_4=0,$$

where u_1, v_1, v_2, u_4

are expressions in x and y. $u_1=0$ is the equation of the flecnodal tangent. The reciprocal polar of a flecnode is a double tangent, which has a simple contact at one point with the reciprocal curve and touches it at a cusp at the other.

(*b*) A BIFLECNODE: It is a node at which both the tangents are stationary tangents. Such a point may be regarded as arising from the union of two points of inflexion with a node. A biflecnode has properties analogous to those of points of inflexion on a cubic. The general equation of a quartic having a biflecnode at the origin is—

$$u_2v_0+u_2v_1+u_4=0.$$

The reciprocal polar of a biflecnode is a pair of cusps having a common cuspidal tangent.

(7) UNDULATION: There is another kind of singular points which a quartic curve can possess. A line may meet a quartic in four points, and when they are consecutive points on the curve, it is called *a point of undulation.*

A point of undulation is defined as a point where the tangent has a contact of the third order with the curve.

In fact, as we shall see later on, a point of undulation is formed by the union of two points of inflexion as consecutive points on a curve. The tangent at such a point is equivalent to two inflexional tangents and one ordinary double tangent. It will be shown in § 135, that the equation of a quartic can be put into the form $\alpha\beta\gamma\delta = S^2$, where α, β, γ, δ touch the quartic at the two points where each intersects the conic S. If now α, β, γ, δ are tangents to S, then α, β, γ, δ are each a tangent to the quartic which has a contact of the third order, *i.e.*, these points of contact will be points of undulation. Thus a quartic has only *four* real points of undulation.

121. When α, β, γ, δ are tangents at points of undulation on a quartic, the conic S is inscribed in the quadrilateral. Four triangles can be formed by taking any three of these four tangents, and any of these triangles can be taken as the triangle of reference, of which S is the inscribed conic. But it is proved in Conics* that the three lines joining the vertices of any triangle circumscribing a conic to the points of contact of the opposite sides meet in a point. Hence we obtain the theorem :—

If a triangle is formed by the tangents at any three real points of undulation, the lines joining the vertices of the triangle with the points of contact of the opposite sides meet in a point.

* Salmon—Conics, § 129.

122. THEOREM :

If a quartic has three points of undulation lying on a right line, the fourth point where the line meets the curve again is a point of undulation.

This is easily proved by the principle of residuation :—

If A, B, C be three collinear undulations and D the fourth point where ABC meets the curve again, we have

$$[\,A+B+C+D\,]=0.$$

Also $\quad [\,4A\,]=0, \quad [\,4B\,]=0, \quad [\,4C\,]=0,$

whence the theorem is proved.

123. THEOREM :

The locus of points, whose first polars break up into a conic and a right line, passes through every point, where the tangent has a contact of the third order with the quartic.

The equation of a quartic which passes through the origin is—

$$u_1+u_2+u_3+u_4=0,$$

where u_1 is the tangent at the origin. If this tangent has a contact of the third order with the curve, u_1 must be a factor in each of u_2 and u_3; consequently the equation of the curve becomes—

$$u_1(v_0+v_1+v_2)+u_4=0.$$

Now, the first polar of the origin is *—

$$u_1(3v_0+2v_1+v_2)=0,$$

which evidently breaks up into the tangent u_1 and a conic, whence the theorem follows.

* We have seen that the first polar of any point passes through the double points on the curve. Hence the first polar of a flecnode breaks up into a line and a conic.

124. Harmonic property:

Any line drawn through a double point on a quartic is harmonically divided by the curve and the first polar of the double point.

Since a double point on a curve is also a double point on the first polar of the same point, any line drawn through this double point meets the quartic in two other points and the first polar only in one other point.

The equation of a quartic having a double point at the origin O is—

$$u_2+u_3+u_4=0 \qquad \dots \quad (1)$$

and the equation of the first polar of O is—

$$2u_2+u_3=0 \qquad \dots \quad (2)$$

Transforming to polar co-ordinates, the equations (1) and (2) become—

$$r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)+r^3(a'\cos^3\theta$$
$$+3b'\cos^2\theta\sin\theta+3c'\cos\theta\sin^2\theta+d'\sin^3\theta)$$
$$+r^4(a''\cos^4\theta+\dots\dots)=0 \qquad \dots \quad (1)$$

and $$2r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)$$
$$+r^3(a'\cos^3\theta+3b'\cos^2\theta\sin\theta$$
$$+3c'\cos\theta\sin^2\theta+d'\sin^3\theta)=0 \quad \dots \quad (2)$$

Now, if P and Q be the points in which a radius vector meets the quartic, and R the point where it meets the first polar, we obtain from equations (1) and (2)—

$$\frac{OP+OQ}{OP.OQ}=\frac{1}{OP}+\frac{1}{OQ}$$

$$=\frac{-(a'\cos^3\theta+3b'\cos^2\theta\sin\theta+3c'\cos\theta\sin^2\theta+d'\sin^3\theta)}{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta}$$

$$=\frac{2}{OR} \quad \text{[by equation (2)]}$$

$\therefore \quad \dfrac{1}{OP}+\dfrac{1}{OQ}=\dfrac{2}{OR}$, *i.e.*, (OPRQ) is a harmonic range.

Ex. 1. Shew that the equation of a quartic with a tacnode at C and the side CA as the tacnodal tangent can be put into the form—

$$y^2z^2+2yzu_2+u_4=0$$

[Use Newton's diagram.]

Ex. 2. Shew that the points of contact of the four tangents to a quartic from a tacnode lie on a conic touching the quartic at the tacnode.

Ex. 3. If a quartic has a rhamphoid cusp at C, the points of contact of the tangents drawn from C lie on a conic osculating the curve at C.

Ex. 4. The line joining two undulations of a quartic meets the curve again in P and Q. Shew that a conic can be described having four-point contact with the curve at P and Q.

Ex. 5. Show how the properties of a quartic with a triple point can be deduced from those of a unicursal cubic.

[Apply a quadric inversion to the quartic with the triple point and any two other points on the curve as the vertices of the fundamental triangle. See § 207, Vol. 1, Theory of Plane Curves.]

125. THEOREM: *A quartic curve cannot have more than two flecnodes.*

The equation of a quartic having three nodes at the vertices A, B, C of the fundamental triangle is—

$$ay^2z^2+bz^2x^2+cx^2y^2+xyz(lx+my+nz)=0.$$

If A is a flecnode, the co-efficients of x^2 and x should have a common linear factor. This requires that $yz(my+nz)$ should contain with $lyz+bz^2+cy^2$ a common linear factor, *i.e.*,

$$lyz+bz^2+cy^2\equiv(my+nz)(m'y+n'z)$$

$$\therefore \quad mm'=c, \quad nn'=b, \quad \text{and} \quad mn'+m'n=l;$$

or $$bm^2+cn^2=lmn \qquad \dots \quad (1)$$

Similarly, if B is a flecnode, we must have—

$$al^2+cn^2=lmn \qquad \dots \quad (2)$$

If now the third vertex C is a flecnode, we must have in addition—

$$al^2+bm^2=lmn \qquad \ldots \quad (3)$$

Thus, from (1), (2), (3) we obtain—

$$al^2+bm^2=bm^2+cn^2=cn^2+al^2=lmn,$$

which give—

$$al^2=bm^2=cn^2=\tfrac{1}{2}lmn=4abc.$$

But these values of the constants reduce the equation to a perfect square, and therefore the third vertex cannot be a flecnode.

126. Theorem: *If a trinodal quartic has two biflecnodes, the third node must also be a biflecnode.*

The equation of a quartic having three nodes at A, B, C is—

$$ay^2z^2+bz^2x^2+cx^2y^2+xyz(lx+my+nz)=0.$$

If A is a biflecnode, the co-efficient of x should contain the co-efficient of x^2 as a factor, *i.e.*, $yz(my+nz)$ should contain bz^2+cy^2+lyz as a factor. But this requires that third powers of y or z should occur, which is impossible since B and C are also nodes on the curve. Hence the only possible conclusion is that $l=m=n=0$; and thus the co-efficient of x vanishes. Hence the equation of a quartic having biflecnodes at A and B is—

$$\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}=0.$$

The symmetry in the result shows that the third vertex C is also a biflecnode.

In order that the curve may be real, one of the constants must have a negative sign (say c). Hence it follows that the nodal tangents at A and B are real, while those at C are imaginary.

127. HARMONIC PROPERTIES OF A BIFLECNODE:

A biflecnode on a quartic possesses harmonic properties analogous to those possessed by a point of inflexion on a cubic. The equation of a quartic having a biflecnode at the origin O is—

$$u_2v_0+u_3v_1+u_4=0 \qquad \ldots \quad (1)$$

The first polar of O is—

$$u_2(2v_0+v_1)=0 \qquad \ldots \quad (2)$$

Hence the first polar of a biflecnode consists of the two biflecnodal tangents and another line which is called the *harmonic polar* of the biflecnode.

Consider a line drawn through the biflecnode at the origin. Let

$$u_2\equiv ax^2+2hxy+by^2, \quad v_1\equiv lx+my;$$

$$u_4\equiv a'x^4+4b'x^3y+6c'x^2y^2+4d'xy^3+e'y^4.$$

Transforming to polar co-ordinates, the equation (1) becomes—

$$r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)(v_0+rl\cos\theta$$
$$+rm\sin\theta)+r^4(a'\cos^4\theta+\ldots\quad)=0.$$

If P and Q be the points in which the radius vector intersects the curve again, then

$$\frac{\mathrm{OP}+\mathrm{OQ}}{\mathrm{OP.OQ}}=\frac{1}{\mathrm{OP}}+\frac{1}{\mathrm{OQ}}=\frac{-(l\cos\theta+m\sin\theta)}{v_0}$$

If R be the point in which the radius cuts the line $2v_0+v_1=0$, we have

$$\frac{1}{\mathrm{OR}}=\frac{-(l\cos\theta+m\sin\theta)}{2v_0}$$

$$\therefore \quad \frac{1}{\mathrm{OP}}+\frac{1}{\mathrm{OQ}}=-\frac{l\cos\theta+m\sin\theta}{v_0}=\frac{2}{\mathrm{OR}}$$

which shows that OR is the harmonic mean between OP and OQ. Hence we obtain the theorem:—

Every line drawn through a biflecnode on a quartic is divided harmonically by the curve and the harmonic polar.

128. THEOREM:

If two right lines be drawn from a biflecnode to meet a quartic in four points and their extremities be joined directly and transversely, the points of intersection will lie on the harmonic polar.

This follows immediately from the harmonic properties of a complete quadrilateral. The method of proof is exactly similar to that used in § 33 in the case of a cubic curve. In exactly the same manner, we may deduce the following—*The tangents at the extremities of any chord drawn through a biflecnode intersect on the harmonic polar.*

129. THEOREM: *The harmonic polar passes through every double point on a quartic.*

This follows as a particular case of the theorem in § 127. For, if O is a biflecnode, and P a double point on the quartic, the line OP cannot meet the curve in any other point. But OP is divided harmonically by the curve and the harmonic polar. Hence, the point where OP meets the harmonic polar coincides with P. Therefore P lies on the harmonic polar.

Again, since a quartic may have two other double points besides the biflecnode, the line joining these double points is the harmonic polar of the biflecnode.

Ex. 1. Shew that a quartic with a biflecnode at O can be put into the form—

$$xyz^2 + u_4 = 0 \qquad \text{or} \qquad (x^2 + y^2)z^2 + u_4 = 0$$

Ex. 2. Shew that the equation of a quartic with two real biflecnodes at A and B can be put into the form—

$$(x^2 + pz^2)(y^2 + qz^2) + 2kz^4 = 0.$$

Ex. 3. Shew that the equation of a quartic with a flecnode at O can be put into form—

$$xyz^2 + 2yzu_2 + x^2v_2 = 0,$$

where u_2 and v_2 are quadratics in x and y.

Ex. 4. Use *Ex.* 1 to show that the points of contact of tangents drawn from O are collinear.

Ex. 5. Prove that a quartic cannot have a biflecnode and a flecnode.

Ex. 6. A quartic cannot have a cusp and a biflecnode.

130. Let us consider the equation $UW=V^2$, where U, V, W each represents a conic. This equation implicitly contains sixteen constants, but the equation of a quartic contains 14 independent constants and therefore the equation of any quartic can be reduced to this form in a doubly infinite number of ways.

Let $$ax^4+by^4+cz^4\ldots\ldots=0 \qquad \ldots \quad (1)$$

be the equation of a quartic. If this is to reduce to the form $UW=V^2$, the equation (1) is to be identified with

$$(x^2+b'y^2+c'z^2+\ldots)(x^2+b''y^2+c''z^2+\ldots)$$
$$-(a_1x^2+b_1y^2+c_1z^2+\ldots)^2=0 \quad \ldots \quad (2)$$

which contains sixteen constants.

If the expanded form of (2) is—

$$Ax^4+By^4+Cz^4+\cdots=0 \qquad \ldots \quad (3)$$

comparing the co-efficients of (1) and (3), we obtain

$$\frac{a}{A}=\frac{b}{B}=\frac{c}{C}=\ \ldots \qquad \ldots \quad (4)$$

There are 14 conditions in the equations (4). Thus the 16 constants must satisfy 14 conditions, and therefore the equation can be reduced in a doubly infinite number of ways.

The form of the equation shows that the conics U and W each intersects the conic $V=0$ in four points lying on the quartic, and each of them cuts a contiguous conic $V'=0$ in four contiguous points on the quartic. Hence the conics U and W each touch the quartic in four points where they respectively meet $V=0$.

131. SYSTEM OF GENERATING CONICS: *

From what has been said above, a quartic can be regarded as the envelope of the singly infinite system of conics represented by the equation

$$\lambda^2 U + 2\lambda V + W = 0 \qquad \ldots \quad (1)$$

where λ is a variable parameter, and $U=0$, $V=0$, $W=0$ represent *three* conics. The conics U and W belong to the system (1) but not V. The equation of the quartic is obtained in the form

$$UW = V^2 \qquad \ldots \quad (2)$$

A conic consecutive to (1), given by—

$$(\lambda+\delta\lambda)^2 U + 2(\lambda+\delta\lambda)V + W = 0$$

intersects (1) in four points which are the points of contact with the envelope.

Hence the conic (1) touches the quartic (2), where $\lambda U + V = 0$, and consequently $\lambda V + W = 0$, *i.e.*, in the four points determined by $\lambda U + V = 0$ and $\lambda V + W = 0$. Thus every conic of the family (1) has a quadruple contact with the quartic and the eight points of contact of U and W lie on the conic V.

The discriminant of the form—

$$\lambda^2 U + 2\lambda V + W$$

is a function of the third degree in its co-efficients and therefore will involve λ in the sixth degree. Thus, six values of λ can be determined so that this discriminant vanishes; and consequently there are six conics of the system (1) which reduce to a pair of right lines. Now each of these pairs of right lines will touch the quartic in four points. But since a line cannot meet a quartic in more than *four* points, each line of a pair touches the quartic at two distinct points, *i.e.*, each line is a *bi-tangent* of the quartic. Thus there are obtained six pairs of bi-tangents of the quartic.

* Plücker—Theorie der Alg. Kurven, p. 228 (Bonn, 1839).

132. The system of generating conics (1) is included in the linear net—

$$\xi U + \eta V + W = 0$$

where ξ and η are variable parameters.

But it is known that a net of conics may always be regarded as the net of first polars of a cubic curve ϕ (say). Then ξ, η may be regarded as non-homogeneous projective co-ordinates of the pole of the polar conic.

If we put $\xi = \lambda^2$, $\eta = 2\lambda$, we obtain the equation

$$4\xi - \eta^2 = 0$$

of a conic ψ of the net, to the points of which the system of conics are associated as polar conics. The quartic therefore may be regarded as the envelope of the first polars of all points of ψ with respect to ϕ, or as the locus of a point, whose polar line with respect to ϕ touches the conic ψ; or finally, as the locus of the poles of tangents of ψ *w.r.t.* ϕ, and this has already been given as one method of generating the quartic.

If S_1 and S_2 are two conics of the system, *i.e.*, polars of two points P_1 and P_2 on ψ, the conic, which passes through the eight points of contact with the quartic, is the polar *w.r.t.* ϕ of the intersection of the tangents to ψ at P_1 and P_2.

The curves ψ and ϕ are called "*Leitkurven*" or the directing curves and the conics of the system *generating conics* of the quartic.

Again, the polar conics of points on a line with respect to a curve of the fourth order envelope a second quartic, and in this way from a given quartic can be derived a second quartic. With the help of this theorem, it can be easily established that the six points on the Steinerian, which correspond to six collinear points on the Hessian, lie on one and the same conic. For further discussion on the subject, see Kohn—Crelle, Bd. 107, p. 5 (1891).

133. SYSTEM OF GENERATING CUBICS:

There are 64 triply infinite systems of curves of the third order which envelope a given curve of order four, *i.e.*, which have six-pointic contact with the quartic.* They belong to two different classes. The first, 36 in number, contains a singly infinite system of degenerate cubics, consisting of lines and conics, in each case the line touches the quartic *once*, and the conic touches it *thrice* and goes through the two remaining points where the line meets the quartic. By means of Hesse's space-representation, the system may be made to correspond to points in space.

The second system, 28 in number, are determined by the bitangents of the quartic. If a bitangent is taken as the side $x=0$ of the fundamental triangle, then the equation of the quartic may be written as $x\phi_3=\phi_2^2$, where ϕ_3 is a cubic touching the curve, and belongs to the system determined by the bitangent, $\phi_2=0$ is the equation of a conic, which passes through the points of contact of the bitangent and also through the six points where the curve ϕ_3 touches the quartic. The twelve points of contact of two enveloping curves of one and the same system, whether it is of the first or the second class, again lie on a curve of the third order. There are 64 curves in each system of enveloping curves, which have four-pointic contact with the quartic.

Two curves of the third order belong to the first or the second class, according as they have a common conic having *triple* contact or *three* collinear points of intersection.†

Ex. A line is drawn through each of the points of contact of a bitangent of a quartic. Show that a cubic touches the quartic at the six points in which these two lines meet the curve again.

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* Hesse—*Crelle*, Bd. 49, p. 243 (1855). Clebsch—*Math. Annalen*, Bd. 3, p. 45 (1871).

† Rosati—*Giorn di Mat.*, Vol. 38, p. 165 (1900).

134. Circuits of a quartic:

The part of a curve traced out by a real point has been called its circuit and is said to be *odd* or *even* according as it is met by any line in an odd or even number of points.* A non-singular quartic therefore cannot have an odd circuit, nor can it have more than four even circuits. For, if it had five, the conic through a point on each circuit would meet the quartic in ten points.

Again, a non-singular quartic can be projected into a closed curve, consisting of one, two, three or four ovals.

Klein's theorem † gives for a non-singular quartic, that if there is no inflexion, there are four ideal bitangents, none meeting the curve in real points. If there is an inflexion, it can be shown by projecting the inflexional tangent to infinity, that there is some line which meets the curve in no real point. In any case, there is some line not meeting the curve in real points, whence the theorem follows.

If there are two ovals, one may lie inside the other, or may be external to it. In the first case, the inner oval can have no inflexion, for in that case the inflexional tangent meets the inner oval in four points and the outer oval in two points at least. But this is not possible for a curve of order four.

In case of three or four ovals, no oval can lie inside another. For in that case a line may be found to meet the quartic in more than four points. If two ovals are external to each other, they will have four and only four common tangents which are bitangents of the curve.

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* Zeuthen—*Math. Ann.* Bd. 7, p. 410 (1874). See also Hilton—Plane Alg. Curves, Chapter XX.

† F. Klein—*Math. Ann.* Bd. 10 (1876), p. 199.

CHAPTER VII

THEORY OF BITANGENTS

135. The bitangents of a quartic have been studied in some details by workers in geometry. The configuration of 28 real bitangents was first discussed by Steiner and Plücker. In this chapter, we shall consider the several methods, but it is convenient to prove at the outset the following important theorem;

THEOREM : *The points of contact of any four bitangents of a quartic lie on a conic.*

The equation $UW = V^2$ can be written in the form—

$$(\lambda^2U + 2\lambda V + W)(\mu^2U + 2\mu V + W)$$
$$= \{\lambda\mu U + (\lambda + \mu)V + W\}^2 \quad \ldots \quad (1)$$

as can easily be verified by multiplying out both sides of the equation, where λ and μ are arbitrary parameters. From what has been said in a previous article it follows that both the conics, represented by the factors on the left, touch the quartic at the four points, where each meets the conic—

$$\lambda\mu U + (\lambda + \mu)V + W = 0.$$

Hence the two sets of four points at which any two of the enveloping conics touch the quartic lie on another conic.

Again, if we determine λ and μ such that

$$\lambda^2U + 2\lambda V + W \quad \text{and} \quad \mu^2U + 2\mu V + W$$

each represents a pair of right lines, the equation (1) takes the form—

$$\alpha\beta\gamma\delta = S^2 \qquad \dots \quad (2)$$

where $$S \equiv \lambda\mu U + (\lambda + \mu)V + W,$$

in which the above values of λ and μ have been substituted. From the form of the equation (2) it follows that each of the lines α, β, γ, δ touches the quartic at two points, where it meets the conic $S=0$, and therefore is a bitangent to the quartic, and the eight points of contact of these four bi-tangents lie on the conic $S=0$.

Notice that the reduction of the equation to the form (2) can be effected in only $\frac{5 \times 6}{2} = 15$ different ways. For λ and μ have each six different values and when we give any value to λ, we must give to μ any one of the *five* remaining values.

136. Configuration of the bitangents:

We have just now shown that the enveloping conics of a quartic and the conics through the points of contact of any pair of them belong to the system included in the equation—

$$lU + mV + nW = 0.$$

But if $$lU + mV + nW = 0$$

represents a pair of right lines, their intersection lies on the Jacobian of U, V, W.* Hence it follows that if the points of contact of an enveloping conic are joined by three pairs of right lines, the intersection of each pair lies on the Jacobian of U, V, W. But this Jacobian is the Hessian of a certain cubic of which

$$lU + mV + nW = 0$$

is the net of polar conics. Therefore each of these lines touches the Cayleyan of the cubic.

* Salmon—Conics, § 388.

Again, since the six pairs of bitangents are included in the form—

$$\lambda^2 U + 2\lambda V + W,$$

it follows that these bitangents all touch the Cayleyan, and the intersection of each pair lies on the Jacobian of the system. It further appears that the lines which join directly or transversely the points of contact of any pair of these bitangents all touch the Cayleyan, and they intersect on the Jacobian.

137. SALMON'S THEOREM: *

Through the four points of contact of any two bitangents, five conics can be described, each of which passes through the four points of contact of two other bitangents.

The equation of a quartic may be written as $UW = V^2$. If now W breaks up into two linear factors α, β, we have

$$\alpha\beta . U = V^2 \quad \ldots \quad (1)$$

and α, β are bitangents of the quartic. This equation can again be written as—

$$\alpha\beta(\lambda^2 U + 2\lambda V + \alpha\beta) = (\alpha\beta + \lambda V)^2 \quad \ldots \quad (2)$$

Now, the discriminant of—

$$\lambda^2 U + 2\lambda V + \alpha\beta$$

is of the sixth degree in λ and contains λ as a factor. The value $\lambda = 0$ corresponds to the bitangents α and β. The other five values of λ, for which the discriminant vanishes, will reduce

$$\lambda^2 U + 2\lambda V + \alpha\beta$$

to a pair of right lines.

Therefore there are *five* different ways in which the equation (1) can be reduced to the form—

$$\alpha\beta\gamma\delta = (\alpha\beta + \lambda V)^2 \quad \ldots \quad (3)$$

* Salmon—H. P. Curves, § 255.

These conics pass through the points of contact of the four bitangents α, β, γ, δ, which proves the theorem.

Since λ can be chosen in five ways, there are six pairs of bitangents, namely $\alpha=0$, $\beta=0$, and five pairs such as $\gamma=0$, $\delta=0$.

A non-singular quartic has 28 bitangents, and there are therefore $\frac{1}{2}.28.27$ or 378 pairs of bitangents. Each of these pairs gives rise to five different conics, but each conic may arise from any one of the six different pairs formed by the four bitangents which correspond to that conic. Hence there are in all $\frac{5}{6}\times 378$ or 315 conics, each of which passes through the points of contact of four bitangents of a quartic.*

138. The equation (2) of the last article enables us to deduce a number of theorems on the points of contact of bitangents. Thus, a conic through the points of contact of two bitangents meets the curve again in four points where a second conic touches the quartic.

For the first conic being $\alpha\beta+\lambda V=0$, the second conic is—

$$\lambda^2 U+2\lambda V+\alpha\beta=0:$$

Similarly, the eight other intersections with the quartic of two conics through the points of contact of two bitangents lie on a conic.

For, if we take—

$$\alpha\beta+\lambda_1 V=0 \quad \text{and} \quad \alpha\beta+\lambda_2 V=0$$

as the conics, the required conic is—

$$\lambda_1\lambda_2 U+(\lambda_1+\lambda_2)V+\alpha\beta=0.$$

* This investigation is due to Salmon. Other workers have discussed the system of these conics—Cayley—Coll. Papers, Vol. VII, p. 123. Hesse—*Crelle*, Bd. 49, p. 243.

139. Steiner-Plücker's Method:

The configuration of the bitangents of a quartic, determined by the nets of conics as stated before, was first investigated by Steiner and Plücker.

The equation of the quartic is put into the form—

$$\alpha\beta\gamma\delta = V^2$$

where α, β, γ, δ are any four bitangents. He states that there are 315 conics, such as V, on which lie by four the points of contact of the 28 bitangents and that each net contains six bitangents and the six intersections of the individual pairs lie on a conic to which all the conics of the series are *apolar.** Thus the 378 points of intersection of the bitangents are said to lie by six on 63 conics to which the nets are apolar. But Plücker does not show how the six points of contact of any three bitangents lie on a conic, and he draws an erroneous conclusion, as pointed out by Salmon, with regard to the number of conics passing through eight points of contact of bitangents.

Steiner's Group: The six pairs of bitangents belonging to a net such that the points of contact of any two pairs lie on a conic are said to form a Steiner's Group, or Set, or Double Six, or a Steiner's Complex.

Definition: Three bitangents of a quartic are called *syzygetic* or *asyzygetic*, according as their six points of contact do or do not lie on a conic.

Four bitangents are said to be *syzygetic* or *asyzygetic* according as their eight points of contact do or do not lie on a conic.

For fuller discussion the student is referred to Frobenius—*Crelle* Bd. 103, p. 169 (1888), and Geiser—*Crelle* Bd. 72, p. 370 (1872), Kohn—*Crelle* Bd. 107, p. 1 (1890).

* A curve of order n and a curve of class n are said to be *apolar* or harmonic, if their mixed bilinear invariant called the *harmonizant* vanishes. See Battaglini—*Napoli Atti.*, Vol. 4, p. 1 (1868).

140. We shall now prove that the equation of the quartic may be reduced to the form—

$$\sqrt{ab}+\sqrt{cd}+\sqrt{ef}=0$$

where a, b, c, d, e, f are three pairs of bitangents. This form of the equation is very convenient for studying the configuration of bitangents, specially the properties of Steiner's Groups. From equation (3) § 137, we may write the equation of the quartic $f=0$ in either of the forms—

$$f\equiv abcd-V^2=0$$

and $$f\equiv abef-W^2=0 \qquad \ldots \quad (4)$$

whence we obtain—

$$ab(cd-ef)\equiv(V^2-W^2)\equiv(V+W)(V-W)$$

Since a, b meet the quartic only in their points of contact, the quartic cannot pass through the intersection of a, b. Consequently, a cannot be a factor of $V+W$ and b of $V-W$, or *vice versa*. For then.

$$V\equiv\tfrac{1}{2}(V+W)+\tfrac{1}{2}(V-W)$$

vanishes, when $a=0$ and $b=0$; and this shows that the curve passes through the intersection of a and b.

Hence we must have—

$$ab\equiv k(V\mp W) \text{ and } cd-ef=(V\pm W)/k$$

where, putting $k=1$, we obtain—

$$-4f\equiv-4abcd+4V^2\equiv-4abcd+(ab+cd-ef)^2$$
$$\equiv a^2b^2+c^2d^2+e^2f^2-2abcd-2abef-2cdef$$

whence the equation of the quartic may be written in the form—

$$\sqrt{ab}+\sqrt{cd}+\sqrt{ef}=0 \qquad \ldots \quad (5)$$

The symmetry in this result shows that the eight points of contact of $c=0$, $d=0$, $e=0$, $f=0$ lie on a conic, whence we obtain the theorem :

The eight points of contact of any two pairs of bitangents lie on a conic.

Ex. 1. Show that the 378 intersections of the 28 bitangents lie by threes on straight lines.

[This may be proved by comparing the two forms of V in the article.]

Ex. 2. Three or four concurrent bitangents are *syzygetic.*

[Let three bitangents be given by—

$$x^2+2hxy+y^2=0 \quad \text{and} \quad y=0.$$

The equation of the quartic is—

$$(x^2+2hxy+y^2)\mathrm{U}=\mathrm{V}^2\,]$$

141. Properties of a Steiner's Group.

Two Steiner's groups are said to be *syzygetic* or *asyzygetic* according as they have four or six common bitangents. The twelve bitangents of a Steiner's group possess a number of important properties. We shall now show that *each bitangent of any pair of a group is cut by the other five pairs in an involution to which the points of contact of the bitangent belong.*

Denote the bitangents of the group by the letters—

$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i,\ j,\ k,\ l.$$

Taking the bitangents k, l as the lines $x=0$, $y=0$, the equation of the curve may be written as—

$$xy\mathrm{U}-\mathrm{V}^2\equiv xy(\mathrm{U}+2k\mathrm{V}+k^2xy)-(\mathrm{V}+kxy)^2=0$$

Hence, the line-pairs ab, cd, ef, gh, ij, are included among the conics—

$$\mathrm{U}+2k\mathrm{V}+k^2xy=0$$

which are cut by $x=0$ in an involution.

Again, *the twelve bitangents of a group touch the same curve of the third class.*

For, the envelope of a line divided by three conics in an involution is a curve of the third class, touching the conics.

This curve is in fact the Cayleyan of the net and is again touched by the lines joining directly and transversely the

points of contact of each pair, *i.e.*, the remaining sides of the six complete quadrilaterals formed by the points of contact of any pair of bitangents. The intersections of each pair of bitangents, and also those of each pair of joining lines, lie on a cubic, which is the Jacobian of the net. There are twelve conics corresponding to each group, each of which touches the quartic twice with ordinary contact, and once so as to meet it in four consecutive points, the twelve points of higher contact lying on the cubic last mentioned.

142. HESSE'S METHOD:

Hesse's method * of studying the configuration of bitangents and the associated conics introduces considerations from the geometry of three dimensions, and was minutely discussed by Prof. Cayley,† and by Nöther.‡ He connects the theory of bitangents with that of the *bundles* of surfaces of the second order. If U, V, W are any three quadric surfaces, then the bundle is represented by the equation—

$$xU+yV+zW=0 \qquad \dots \quad (1)$$

The cones belonging to this bundle are determined by the values of x, y, z, obtained by equating to zero the discriminant of (1). The discriminant is of the fourth order, and represents a curve of the fourth order, if x, y, z are regarded as homogeneous point co-ordinates in a plane. Thus there is established a (1, 1) correspondence between this quartic and the space-curve of the sixth order through the vertices of the cones. In fact, for each point of the quartic a system of values of x, y, z is determined for which the discriminant vanishes, and consequently this gives a cone in the bundle or the vertices of the cones. On the other hand, corresponding to a cone in the bundle,

* Hesse—*Crelle*, Bd. 49, p. 243 (1855)

† Cayley—*Crelle*, Bd. 68, p. 176 (1868).

‡ Nöther—*Math. Ann.* Bd 15, p. 89 (1879).

a system of values of x, y, z is obtained for which the discriminant vanishes and a point of the quartic is obtained.

A linear relation between x, y, z represents a line in the plane and a fixed pencil in the bundle of surfaces. To the four points in which the line cuts the curve correspond the vertices of the four cones contained in the pencil. If the line is a bitangent, then the basis-curve of the pencil consists of a space-curve of order three and one of the double lines through the vertices; the space-curve passes through six of the eight points common to the surfaces of the bundle; while the line joins the remaining two points. Thus the bitangents of the quartic are placed in a simple relation to the 28 lines joining the eight points of the pencil. If we denote the eight points in space by means of the numerals 1 to 8, to each combination of two of the numbers 1 to 8, there corresponds one bitangent of the quartic.

Thus Hesse introduces a notation by means of which the configuration of the 28 bitangents and the associated conics can be easily determined.

The 378 pairs of bitangents will be represented by the two similar types of symbols (12. 13), etc., and (12. 34), etc. There are then 1,260 triads of bitangents, whose six points of contact lie on a conic and they belong to two different types (12. 23. 34) etc. and (12. 34. 56) etc.

There are 315 tetrads of bitangents whose eight points of contact lie on a conic, and belong to two different types (12. 23. 34. 41) etc., and (12. 34. 56. 78) etc.

There are 1,008 groups of six bitangents which touch a conic and they may be represented by one of the following three types of symbols (12. 23. 31. 45. 56. 64), (12. 34. 35. 36. 37. 38), (12. 13. 14. 56. 57. 58).

There are 5,040 groups of six bitangents which intersect in pairs in three collinear points and they may be represented by one of the following types: (12. 23. 31. 14. 45. 51), (12. 23. 34. 45. 56. 61), (12. 34. 35. 36. 67. 68).

143. Application of Hesse's notation:

Hesse's notation, although not symmetrical, is perhaps as convenient as any we can find.

Combination in pairs of the eight symbols 1, 2, 3, 4, 5, 6, 7, 8, gives us 28 symbols 12, 13,...78, which may be used to denote the 28 bitangents. The notation serves to correctly represent the geometrical relations of the 28 bitangents. For the notation might suggest that the bitangent 12 was related in a different manner to the bitangents 13, 14, etc., and to the bitangents 34, 56, etc., but there is no real geometric difference between the relations of any pair of bitangents.

We again apply the symbol such that 12, 34, 56, 78 denote the bitangents whose eight points of contact lie on a conic. The same property will then belong to every tetrad of bitangents represented by a similar set of duads, *i.e.*, by any four duads containing all the eight symbols. But it will be found that there are only 105 arrangements of the eight symbols into sets such as 12, 34, 56, 78. The remaining 210 conics correspond to four bitangents represented by the symbols such as 12, 23, 34, 41, *i.e.*, the duads formed cyclically from any arrangement of four of the eight symbols. It is found that there are then 210 such tetrads.

Hence the group belonging to the pair 12, 34 consists of

56, 78; 57, 68; 58, 67; 13, 24; 14, 23;

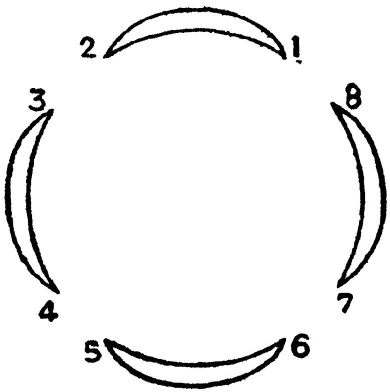
and the group belonging to the pair such as 12, 13 is--

24, 34; 25, 35; 26, 36; 27, 37; 28, 38.

Thus we see that the notation shows completely how the bitangents are to be combined in groups. But although the notation suggests that the first 105 conics differ in their properties from the last 210 conics, this is not actually the case, since the 315 conics are of the same nature.

The notation has been fully discussed by Prof. Cayley.* He exhibits in a table the geometrical relations of the bitangents, taken singly in *twos*, *threes* or *fours*, and the number of terms belonging to each type of arrangement of the symbols. For a detailed study of these symbols, the student is referred to the paper by Prof. Cayley and to Salmon's H. P. Curves §§ 259-262.

144. Fontenè,† however, has given a method of visualizing the notation of Hesse by considering a quartic $fg+\epsilon=0$, where ϵ is a small constant, and $f=0$, $g=0$ are equal ellipses of small eccentricity and nearly concentric, the major axis of one being parallel to the minor axis of the other. The quartic thus consists of four portions



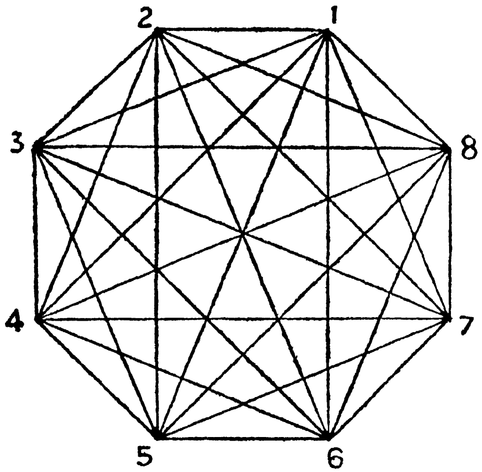
each lying inside one ellipse and outside the other. The portions are narrow ovals each touched by one bitangent, while any two ovals have four common bitangents. The quartic has 28 real bitangents, each touching the curve in points which lie very close to two of the points denoted by the symbols 1, 2, 3, 4, 5, 6, 7, 8, in the figure.

* Cayley—*Crelle*, Bd. 68 (1868).

† G. Fontené—Bull. de la Soc. Math de France, Vol. 27, p. 289 (1899).

See also Hilton—Plane Alg. Curves, p. 342.

Next suppose that the ellipses are nearly equal, so that in the limit the ovals are indefinitely narrow. The bitangents may now be regarded as the 28 lines joining in pairs the eight points 1, 2, 3, 4, 5, 6, 7, 8.



The bitangents are then denoted by 12 or 21, 13 or 31, etc. It is readily seen that three bitangents are *syzygetic* if (1) their symbols involve six distinct digits, namely, 12, 34, 56; or (2) their symbols contain two digits twice and two digits once, namely 12, 23, 34.

Similarly, four bitangents are *syzygetic*, if they can be represented by either of the two symbols of the types

12, 34, 56, 78; or 12, 23, 34, 41.

From the figure it will be easily seen that the four bitangents either join four distinct pairs of points or are the sides of a quadrilateral, whence it follows that there are Groups of the two types—

18 17, 28 27, 38 37, 48 47, 58 57, 68 67;

or 14 23, 24 31, 34 12, 58 67, 68 75, 78 56.

There are then ${}^8C_2$ or 28 Complexes of the first type and $\frac{1}{2}.{}^8C_4$ or 35 Complexes of the second type, altogether making

up the *sixty-three* Complexes, and this is the complete list of possible sets of three or four *syzygetic* bitangents.

This explains the notation used by Hesse and the students are referred for further details to the works cited before.

145. GEISER'S METHOD:

Another mode of connecting the theory of bitangents with space-geometry was used by Geiser.* From any point on a cubic surface can be drawn a quartic cone touching the surface, and this intersects any plane in a general quartic curve. The cone will be non-singular, its bitangent-planes being the tangent plane to the cubic at the vertex and the planes joining the vertex to the 27 lines on the surface. The 28 bitangents of the quartic are the lines in which these planes are met by the plane of the quartic. He thus establishes a relation between the 28 bitangents of a quartic with the 27 lines on a cubic surface.

Zeuthen shows that his classification of quartics with regard to the reality of their bitangents leads by a different process to the results obtained by Schläfli in classifying cubic surfaces with respect to the reality of their right lines.

146. ARONHOLD'S METHOD:

Aronhold has shown that when seven arbitrary lines are given, a quartic curve can be found having these lines as bitangents, and of which the other bitangents can be found by linear construction.

The method depends on properties of a system of curves of the third class having seven common tangents and can be explained briefly as follows:

* Geiser—*Math. Ann.* Bd. 1 (1869), p. 129. See also Zeuthen—*ibid*, Bd. 8 (1875), p. 1.

Let us consider two curves ϕ and ψ of class three, having seven fixed common tangents. They have then two other common tangents, which intersect at any point P (say). If now we keep ϕ fixed and vary ψ in such a way that it always touches the seven given lines, then through P there passes a tangent a of ϕ, which may be called the *coresidual* * of ϕ with respect to the seven given lines. The tangent a intersects this curve ϕ, which is of order 6, in four other points. At these four points, and at no other, will the curve ϕ be touched by any other curve of class 3 having those seven lines as tangents. Hence follows that the locus of the points of contact of the two curves of class three of this system is a curve of order four,† and in fact, is a general quartic having the seven given lines as bitangents. In order to determine the remaining 21 bitangents, we have to find 21 curves of the third class belonging to the system, which break up into a conic and a line. Five of the seven given lines must touch the conic, and the remaining two intersect on the line. The coresidual lines to these 21 degenerate curves are the bitangents sought, which can be obtained by linear construction only.

The set of seven lines, which are themselves bitangents of the quartic, is called an "Aronhold Seven." They are such that the points of contact of any three do not lie on a conic, *i.e.*, no three of these seven are *syzygetic*.

* If we have a system of curves of the third class touching seven given lines, and consider any one curve of the system, the eighth and ninth tangents, common to it with any other curve of the system, intersect on a fixed tangent of the selected curve, which may be called the *coresidual*, for that curve, of the seven given tangents.

† This can be easily seen by considering a system of cubics through seven given points. If the remaining two points coincide, *i.e.*, if the cubics touch, the envelope of the common tangent is a curve of the fourth class. The reciprocal property will establish the theorem. See Salmon—H. P. Curves, §§ 263-64.

147. LINEAR CONSTRUCTION OF BITANGENTS: *

The bitangents of a quartic can accordingly be found from the seven given bitangents by linear constructions. From what has been said above, we have in fact to construct the coresidual tangents for the several systems 12345, 67, etc., where 12345 denotes the conic touching the first five lines, and 67 is the point of intersection of the other two. Now, the two systems 12345 67 and 12346 57 have obviously seven common tangents, and the remaining common tangents are the tangents to 12345 from the point 57, and to 12346 from 67. But when one point on a tangent to a conic is given, the remaining tangent can be obtained by linear constructions with the help of Brianchon's theorem. Thus the two tangents and their intersection are obtained. The remaining tangents drawn from the point to each of the two conics will be the two required coresiduals, and consequently two of the bitangents. The same may be obtained by considering the three systems 12345, 67; 12346, 57; 12347, 56. We determine, as before, the remaining eighth and ninth tangents common to each pair of systems, the three intersections of these pairs of tangents, being joined, will give three of the required bitangents. The bitangent which is the coresidual for the system 12345, 67 may be called the bitangent (67); and thus the 21 bitangents may be denoted by combinations of the symbols 1, 2, 3, 4, 5, 6, 7. In addition to these, there are the seven given lines as bitangents. If now we introduce for symmetry a new symbol 8, the seven lines may be denoted by (18), (28), (38), (48), (58), (68), (78) and this leads to a algorithm identical with that of Hesse.

For a complete discussion of the method, the student is referred to Aronhold's original paper in *Berlin Monatsberichte*—(1864), p. 499; Timerding—*Math. Ann.* Bd. 53 (1900), p. 193.

* Salmon's H. P. Curves, §§ 263-268.

148. METHOD OF CHARACTERISTICS:

In order to investigate the bitangents of a quartic E. Pascal* has introduced a method in which bitangents are represented by means of so-called odd characteristics of genus 3. Each bitangent is represented by the symbol

$$\begin{pmatrix} i & j & k \\ i_1 & j_1 & k_1 \end{pmatrix}$$

where i, j, k, i_1, j_1, k_1 are either 0 or 1, and the sum

$$ii_1+jj_1+kk_1$$

is an odd number (1 or 3). In this representation, the four bitangents, whose points of contact lie on a conic, are represented by four characteristics, whose corresponding elements give an even sum (0 or 2). For a study of the theory, the reader is referred to Nöther—*München Abhandlungen*, Bd. 17 (1879), p. 105, and *Math. Ann.* Bd. 28 (1886), p. 354.

149. INVESTIGATION BY KUMMER'S CONFIGURATION:

Kummer had remarked that ∞^4 of surfaces of order 4 with 16 nodes and 16 singular tangent planes pass through a general quartic. These 16 tangent planes cut out 16 bitangents of the quartic, so that the 12 remaining bitangents form a Steiner's Group.† The 16 of the 28 bitangents, which are obtained excluding the Steiner's Group and which could be determined on the plane of a Kummer's configuration in the above manner, are said to form a Kummer's Group. The 16 bitangents of a

* E. Pascal—*Ann. di Mat.*, Vol. 20 (1893), pp. 163, 269 and Vol. 21 (1893), p. 85.

† In this way the theory of bitangents of Ciani has been developed—*Ann. di Mat.*, Vol. 2 (3) (1897), p. 53.

Kummer's Group touch by six 16 conics. It follows further that the r points, in which two bitangents of a Kummer's Group intersect, lie by threes on 240 lines π, and these lines pass by threes through 1280 points, 320 of those points are obtained as follows:—The 6 points, in which the pairs of bitangents of the Steiner's Group, complementary to the Kummer's Group, intersect, determine a complete hexagon. The points, in which each side of this hexagon is met by the pairs of opposite sides of the complete quadrilateral formed by the four vertices of the hexagon not lying on it, determine an involution, whose double points are denoted by M (say). The 30 points M thus obtained lie by threes on 60 lines m, which form 15 quadrilaterals with vertices M. The 60 lines m pass by threes through 320 points s, which are the required points.

If we consider the 63 Kummer's Group, we may say that the 378 points of intersection of the 28 bitangents lie by threes on 5040 lines π, which again pass by threes through 6720 points s and (0480 points k. Each line π contains four points s and 36 points k. The 6720 points s lie by fours on 15120 lines σ. The 60480 points k lie by threes on 20160 lines c, which pass through the 6720 points s by threes, and through 15120 points i by fours. For other details, see H. Webers—Algebra Bd. 2 (1900).*

Ex. 1. Shew that there are 288 Aronhold Sevens of bitangents, of which 72 contain any given bitangent and 16 contain any two given bitangents.

Ex. 2. Using Aronhold's method shew that the intersections of the bitangents lie, three by three, on straight lines.

[See Salmon—H. P. Curves, § 266].

Ex. 3. Shew that the six intersections of each pair of a Steiner's Group lie on a conic.

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* De Paolis, considering a quartic as an intermediary curve in a plane transformation of order three and deficiency one, obtained the properties and the equations of the 28 bitangents (1878).

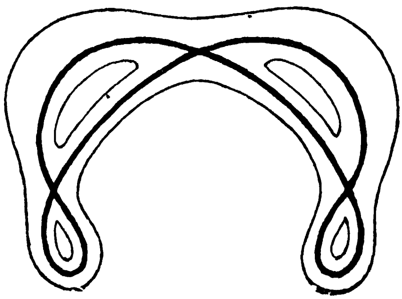
150. THE REAL BITANGENTS:

Plücker * gave the following example for showing that the 28 bitangents of a quartic are all real. In this case the quartic must necessarily be quadripartite. These four branches have therefore ${}^4C_2 \times 4 = 24$ common tangents, which are bitangents of the quartic.

There must be other four bitangents for which Plücker starts with the equation—

$$\Omega \equiv (y^2 - x^2)(x-1)(x-\tfrac{3}{2}) - 2\{y^2 + x(x-2)\}^2 = 0.$$

The circle within the parenthesis is circumscribed about the triangle $(y^2 - x^2)(x-1)$. Therefore the curve has double points at the three vertices, and $x - \frac{3}{2} = 0$ is a bitangent, as shown in the figure by the thick line.



Let us now consider the curve $\Omega = k$, for small integral positive or negative values of k.

The curve $\Omega = k$ does not meet $\Omega = 0$ in any finite point, and it deviates less from the form of the curve Ω, the less we suppose k, and according as k is positive or negative, it is altogether within or without Ω. When k is negative, the curve is altogether without, and it is *unipartite* and

* Plücker—Theorie der Algebraischen Kurven, p. 247.

has 28 real bitangents. When the curve is within, it consists of four ovals, one in each of the compartments into which the curve Ω is divided. Each oval has one tangent touching it doubly. Hence there are four bitangents belonging to the ovals.

Again, any two ovals have four common tangents and the four ovals can be grouped into six such pairs. There are therefore 24 bitangents, and thus altogether there are 28 real bitangents.

Thus, in the above example, there are 4 bitangents of the *first kind* and 24 of the *second kind*, as was otherwise obtained by Zenthen.

In fact, the number of real bitangents of a quartic is 4, 8, 16 or 28.

For, if the quartic has $r(r=1, 2, 3, 4)$ ovals external to each other, it has also $\frac{1}{2}r(r-1)\times 4$ bitangents of the second kind. But it has four bitangents of the first kind. Hence the curve has 0, 4, 12 or 24 bitangents of the second kind and four of the first kind, which proves the proposition.

Ex. 1. Show that the curve $x^3y+y^3z+z^3x=0$ has exactly four real bitangents.

[Tracing the curve it will be seen that it consists of a single oval touched in real points by three bitangents. The line $x+y+z=0$ is the fourth real bitangent.]

Ex. 2. The eight points of contact of the four bitangents of the first kind lie on a conic.

Ex. 3. Three bitangents of a quartic form a triangle ABC, whose sides touch the quartic at P_1 and P_2, Q_1 and Q_2, R_1 and R_2. Show that the conic $P_1Q_1Q_2R_1R_2$ passes through either P_2, or the harmonic conjugate of P_2 *w.r.t.* B and C.

Ex. 4. Two external ovals cannot be enclosed within the triangle formed by any three bitangents of the first kind.

Ex. 5. Two external ovals of a quartic have two bitangents having the ovals on the same side and two bitangents having the ovals on opposite sides.

151. Real inflexions and undulations:

When a tangent touches a single branch at two real points, it is evident that the arc at each of these points is convex towards the tangent. Therefore, intermediate between these two convex parts, there must be a part of the arc which is concave towards the tangent. This concave part must therefore be separated from the convex parts by *a point of inflexion* at each extremity. Therefore, corresponding to each bitangent of the first kind, there are two *real* points of inflexion. Conversely, the existence of two real points of inflexion on a single branch implies the presence of a bitangent of the *first kind*. Now, since there are only four such bitangents, *a quartic curve cannot have more than eight real points of inflexion.*

If, however, the two points of inflexion coincide, the bitangent touches the curve at four consecutive points and the point of contact becomes a *point of undulation*. Hence, since there are only eight real points of inflexion, a quartic can have at the most *four* real points of undulation.

Ex. 1. The tangents at three collinear inflexions of a quartic meet the curve again in collinear points.

Ex. 2. The line joining two undulations of a quartic meets the curve again in P, Q. Show that a conic can be described having four-pointic contact with the curve at P and Q.

Ex. 3. Show that there are six real inflexions on the quartic—

$$x^3y+y^3z+z^3x=0$$

and that they lie on the conic $yz+zx+xy=0$.

CHAPTER VIII

UNICURSAL QUARTICS

152. RATIONAL QUARTICS: *

If a quartic is rational (or unicursal) of deficiency zero, then the co-ordinates of any point can be represented by

$$x : y : z = f_1 : f_2 : f_3$$

where f_1, f_2, f_3 are rational functions of a parameter.

Quartics of species VII, VIII, IX, and X belong to this class, and include besides quartics with a triple point or other kinds of higher singularities equivalent to three double points. While discussing Quadric Inversion, we have seen that a rational quartic can be obtained from a conic by a quadric transformation. Schröter † considered such a quartic as the locus of a point common to the corresponding tangents to two projective conics. When, however, one conic reduces to a point, the locus becomes a rational cubic. Stahl ‡ defined a rational quartic as the plane projection of a rational curve in space of order four.

From the above parametric representation of a unicursal quartic, the Cartesian and tangential equations of the curve, its singularities, collinearity of points, etc., can be easily obtained.§

* For general methods of treatment of rational curves, the student should consult Meyer's *Apolarität und Rationale Kurven.*

† Schröter—*Crelle*, Bd. 54, p. 32 (1857).

‡ Stahl—*Crelle*, Bd. 101, p. 300 (1887).

§ Brill—*Math. Ann.*, Bd. 12, p. 90 (1877). See also—Theory of Plane Curves, Vol. I, Chap. XIII.

A unicursal quartic has four bitangents, six inflexions and is of class six, as can easily be verified by Plücker's formulæ.

Properties of a unicursal quartic are best studied by the method of quadric inversion, but geometrical processes are at times useful and convenient. We may project two of the double points into the circular points at infinity and then invert with respect to the third double point. A conic is thus obtained, from each of whose properties a property of the quartic can be deduced. It is to be noted, however, that this method will be convenient if the two projected double points are of the same kind, but there is no such restriction in the case of quadric transformation.

The properties of a tricuspidal quartic can be easily derived from those of a nodal cubic by reciprocation.

Ex. 1. Show that any quartic with a node and a pair of cusps can be projected into a limaçon.

[Projecting the cusps into the circular points at infinity with the node at the origin, we get a limaçon, which is the inverse of a conic *w.r.t.* a focus.]

Ex. 2. Show that the locus of the intersection of the tangents at corresponding points of homographic ranges on two given conics is a unicursal quartic.

[Take the equation of the corresponding tangents as—

$$\lambda^2U + 2\lambda V + W = 0 \quad \text{and} \quad \lambda^2U' + 2\lambda V' + W' = 0$$

where U, V, W, U', V', W' are linear expressions in x, y, z.]

Ex. 3. Prove that the cuspidal tangents of a tricuspidal quartic meet in a point.

[This may be obtained by reciprocating the theorem: The three inflexions of a nodal cubic are collinear.]

Ex. 4. Show how the properties of quartics belonging to species IX can be deduced from those of a limaçon by projection and reciprocation.

Ex. 5. A pencil of quartics with three given nodes through four other fixed points cuts any line through a node in an involution.

[Applying a quadric inversion, we get a pencil of conics which cuts any line in an involution.]

153. TRINODAL QUARTICS:

We have already seen that the equation of a trinodal quartic having three nodes at the three vertices A, B, C of the triangle of reference can be written as

$$\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}+\frac{2f}{yz}+\frac{2g}{zx}+\frac{2h}{xy}=0 \qquad \ldots \quad (1)$$

or $$ay^2z^2+bz^2x^2+cx^2y^2+2xyz\,(fx+gy+hz)=0 \quad \ldots \quad (2)$$

The tangents at the nodes A, B, C are given respectively by
$$cy^2+bz^2+2fyz=0, \qquad az^2+cx^2+2gzx=0,$$
$$bx^2+ay^2+2hxy=0.$$

The equation (2) can be put into the form

$$(ayz+hzx+gxy)^2+(ca-g^2)x^2y^2$$
$$-2(gh-af)x^2yz+(ab-h^2)z^2x^2=0$$

or $$(ayz+hzx+gxy)^2+x^2(\mathrm{B}y^2-2\mathrm{F}yz+\mathrm{C}z^2)=0$$

where A, B, C, ... denote the co-factors of a, b, c,...etc. in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

This shows that the lines $\mathrm{B}y^2+\mathrm{C}z^2-2\mathrm{F}yz=0$ touch the quartic at the points (other than the nodes) where the conic $ayz+hzx+gxy=0$ meets it. These lines through A therefore represent the tangents which can be drawn to the curve from the node A. Similarly, the equations of the tangents from B and C can be obtained.

Putting λ^2, μ^2, ν^2 for a, b and c respectively, the equation (2) can be written as—

$$\lambda^2y^2z^2+\mu^2z^2x^2+\nu^2x^2y^2+2xyz\,(fx+gy+hz)=0$$

i.e. $$(\lambda yz+\mu zx+\nu xy)^2$$
$$+2xyz\{(f-\mu\nu)x+(g-\nu\lambda)y+(h-\lambda\mu)z\}=0$$

which shows that the line

$$P \equiv (f-\mu\nu)x+(g-\nu\lambda)y+(h-\lambda\mu)z=0 \qquad (3)$$

touches the quartic at the points where the conic

$$\lambda yz+\mu zx+\nu xy=0$$

meets it, *i.e.*, it is a bitangent.

Since the equation of the quartic remains unaltered when the sign of any one of the quantities λ, μ, ν is changed, we obtain the other bitangents by writing $-\lambda$, $-\mu$, $-\nu$ respectively for λ, μ, ν in (3).

Thus the equations of the four bitangents can be written as

$$(f\pm\sqrt{bc})x+(g\pm\sqrt{ca})y+(h\pm\sqrt{ab})z=0$$

where one or all signs are to be taken negatively. If now we write for a moment

$$u\equiv fx+gy+hz,\quad \sqrt{bc}.\,x=\alpha,\quad \sqrt{ca}.\,y=\beta,\quad \sqrt{ab}.\,z=\gamma,$$

the four bitangents (Θ) are represented by

$$\Theta\equiv(u-\alpha-\beta-\gamma)(u-\alpha+\beta+\gamma)(u+\alpha-\beta+\gamma)(u+\alpha+\beta-\gamma)$$

$$=(u^2-\alpha^2-\beta^2-\gamma^2)^2-4(\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2+2\alpha\beta\gamma u)$$

$$=(u^2-\alpha^2-\beta^2-\gamma^2)^2-4abcU.$$

Therefore the equation of the quartic U may be written in the form—

$$4abcU\equiv\{(fx+gy+hz)^2-bcx^2-cay^2-abz^2\}^2-\Theta=0$$

showing that the points of contact of the bitangents lie on the *bitangential* conic.*

$$(fx+gy+hz)^2-(bcx^2+cay^2+abz^2)=0$$

* Theory of Plane Curves, Vol. I, § 148.

Ex. 1. Tangents are drawn to a trinodal quartic from two nodes. Prove that the points of contact lie on a conic through the two nodes.

[Project the nodes into the circular points and invert *w.r.t.* the third node. The inverse is a conic which intersects a pair of directrices in concyclic points. For a direct analytical proof, see Basset—*Cubics and Quartics*, p. 130.]

Ex. 2. Deduce the tangential equation of the trinodal quartic.

[The line $\lambda x+\mu y+\nu z=0$ touches the quartic

$$a/x^2+b/y^2+c/z^2+2f/yz+2g/zx+2h/xy=0$$

when also $$\lambda/x+\mu/y+\nu/z=0$$

and $$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

touch. Hence the tact-invariant of these two conics will give the required tangential equation].

Ex. 3. Show that the diagonals of the quadrilateral formed by the bitangents are

$$\mathrm{H}y+\mathrm{G}z-\mathrm{A}x=0,\quad \mathrm{H}x+\mathrm{F}z-\mathrm{B}y=0 \quad\text{and}\quad \mathrm{G}x+\mathrm{F}y-\mathrm{C}z=0.$$

Ex. 4. Show that the triangle formed by the diagonals is in plane-perspective with the triangle formed by the nodes.

Ex. 5. Show that the triangle formed by any three bitangents is in plane perspective with the triangle formed by the nodes, and the four centres of perspective form a quadrangle with this latter as its harmonic triangle.

[The centres of perspective are $\pm\sqrt{a}$, $\pm\sqrt{b}$, $\pm\sqrt{c}$.]

154. The equation

$$\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s}=0$$

where p, q, r, s are linear expressions in x, y, z.

On rationalising, the equation becomes—

$$(p^2+q^2+r^2+s^2-2pq-2qr-2rs-2ps$$
$$-2pr-2qs)^2=64pqrs.$$

or $$\{2(p^2+q^2+r^2+s^2)-(p+q+r+s)^2\}^2=64pqrs$$

This shows that the given equation represents a quartic curve with p, q, r, s as bitangents, the bitangential conic being

$$2(p^2+q^2+r^2+s^2)-(p+q+r+s)^2=0$$

The given equation can also be written as—

$$[(p-r)+(q-s)]^4-8[(p-r)+(q-s)]^2(pq+rs)$$
$$+16[(p-r)q+(q-s)r]^2=0$$

each term of which contains either

$$(p-r)^2 \quad \text{or} \quad (p-r)(q-s) \quad \text{or} \quad (q-s)^2$$

Hence, the point $p-r=0$, $q-s=0$ is a double point (a node).

Similarly, the points $p-s=0$, $q-r=0$ and $r-s=0$, $p-q=0$ are nodes on the locus.

Therefore the given equation represents a trinodal quartic with p, q, r, s as bitangents. The converse theorem is also true, namely, the equation of a trinodal quartic can be put into the above form.

155. Application of Quadric Inversion.

While discussing the process of quadric inversion, we had occasion to consider some of the properties of a trinodal quartic. We shall now consider some other properties by the application of the same process.

A trinodal quartic has four bitangents and six points of inflexion.

Now, applying the principles of quadric inversion, the bitangents correspond to conics through the nodes A, B, C, having double contact with the conic and the stationary tangents correspond to conics through A, B, C, osculating the adjoint conic. Hence we obtain the theorem :

Through three given points there can be drawn four conics having double contact with a given conic and six conics which osculate a given conic.

Various other properties of the curve can similarly be established; but it is important to carefully note the correspondence of the adjont conic and the quartic derived from it, The conic meets the sides of the triangle in points to which the tangents at the opposite vertices correspond. Hence, according as the conic meets a side in two imaginary points, touches it or meets it in two real points, the quartic has at the opposite vertex an acnode, a cusp or a crunode.

156. Conics associated with a trinodal quartic.

A trinodal quartic possesses six points of inflexion. The six nodal tangents, so also the six inflexional tangents, touch a conic. In fact, there are a number of conics associated with a trinodal quartic which we shall discuss in this article. Ferrers* proved that the six inflexional tangents touch one and the same conic, but left the correlative theorem that the six points of inflexion lie on a conic unconsidered. The complete solution of this and other cognate problems has, however, been given in an exhaustive memoir by Brill.† The memoir is so interesting and useful to students of analysis that we have preferred to give an extract in the form of an appendix at the end of this volume. A direct method of obtaining the actual expression for the inflexions-conic has, however, been recently given by Richmond and Stuart.‡ A number of interesting properties of the trinodal quartic have been discussed by R. A. Roberts, § and among others, from the tangential equation of the quartic he deduces the equation of the conics enveloped by the six inflexional tangents and the nodal tangents.

* Ferrers—*Quarterly Journal*, Vol. 18, p. 73 (1882).

† Brill—*Math. Ann.*, Bd. 12, pp. 98-122 (1877).

‡ Proc. of the London Math. Soc., Vol. 2 (1), p. 130 (1903).

§ Roberts—Proc. Lond. Math. Soc., Vol. 16, pp. 44-60 (1884).

CONIC I: *The six nodal tangents of a trinodal quartic touch one and the same conic.**

Consider the tangents at the node C.

Let $bx^2+ay^2+2hxy\equiv(l_1x+m_1y)(l_2x+m_2y)=0$

whence $m_1/l_1+m_2/l_2=+2h/b$ and $m_1m_2/l_1l_2=a/b$.

Thus the ratios m_1/l_1 and m_2/l_2 are given as the roots of the equation

$$bm^2-2hlm+al^2=0.$$

Similarly, if we find the corresponding equations satisfied by the parameters of the other two pairs of tangents, it will be seen that these six tangents touch the conic

$$al^2+bm^2+cn^2-2fmn-2gnl-2hlm=0 \qquad (1)$$

For, if in this equation we put $n=0$, we obtain the equation satisfied by the parameters of the nodal tangents at C.

Equation (1), when expressed in point co-ordinates, may be written as—

$$(bc-f^2)x^2+(ca-g^2)y^2+(ab-h^2)z^2$$
$$+2(gh+af)yz+2(fh+bg)zx+2(fg+ch)xy=0$$

i.e. $$(\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+2\mathrm{F}yz+2\mathrm{G}zx+2\mathrm{H}xy)$$
$$+4(afyz+bgzx+chxy)=0 \quad \ldots \quad (2)$$

Equation (2) may again be expressed in the form—

$$cy^2+bz^2+2fyz+k^2\{(bc-f^2)x+(fg+ch)y+(fh+bg)z\}^2=0$$

where $k^2\{abc-af^2-bg^2-ch^2-2fgh\}=1.$

Therefore the chord of contact of the nodal tangents at A is represented by the equation

$$(bc-f^2)x+(fg+ch)y+(fh+bg)z=0$$

Other chords can be similarly obtained.

* Theory of Plane Curves, Vol. I, § 216, Ex. 1.

CONIC II: *The six points where the nodal tangents meet the sides of the nodal triangle lie on a conic.*

The intersections of the tangents with the sides are given by—

$$x=0, \quad cy^2+bz^2+2fyz=0$$

$$y=0, \quad az^2+cx^2+2gzx=0$$

$$z=0, \quad bx^2+ay^2+2hxy=0.$$

Hence the conic—

$$bcx^2+cay^2+abz^2+2afyz+2bgzx+2chxy=0$$

passes through the points in question.

CONIC III: *From each node of a trinodal quartic two tangents can be drawn to the curve, and these six tangents touch one and the same conic.**

The equation of the tangents drawn from the nodes are—

$$By^2+Cz^2-2Fyz=0$$

$$Cz^2+Ax^2-2Gzx=0$$

$$Ax^2+By^2-2Hxy=0.$$

Proceeding exactly as in Conic 1, we find that the tangential equation of the conic touched by these tangents is—

$$BCl^2+CAm^2+ABn^2+2A(gh-af)mn$$
$$+2B(fh-bg)nl+2C(fg-ch)lm=0$$

and the point-equation is obtained in the form—

$$A^2ax^2+B^2by^2+C^2cz^2+2fBCyz+2gCAzx+2hABxy=0.$$

* Theory of Plane Curves, Vol. 1, §216, *Ex.* 1.

CONIC IV: *The six tangents drawn from the nodes meet the sides of the nodal triangle in six points lying on a conic.*

These points are given by—

$$x=0,\quad By^2+Cz^2-2Fyz=0$$

$$y=0,\quad Cz^2+Ax^2-2Gzx=0$$

$$z=0,\quad Ax^2+By^2-2Hxy=0.$$

whence they are found to lie on the conic—

$$Ax^2+By^2+Cz^2-2Fyz-2Gzx-2Hxy=0.$$

CONIC V: *The six inflexional tangents of a trinodal quartic touch one and the same conic.*

If the line $\lambda x+\mu y+\nu z=0$ be an inflexional tangent, it has three-point contact with the quartic. If now we apply a quadric transformation, *i.e.*, replace x, y, z by $1/x$, $1/y$, $1/z$ respectively in the equations of the line and the quartic, the resulting conics, namely,

$$S\equiv\lambda yz+\mu zx+\nu xy=0 \quad \dots \quad (1)$$

and $$R\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \dots \quad (2)$$

will have also three-point contact.

But the conditions * that these two conics may have a contact of the second order are given by—

$$3\Delta/\Theta=\Theta/\Theta'=\Theta'/3\Delta' \quad \dots \quad (3)$$

where $$\Delta\equiv abc+2fgh-af^2-bg^2-ch^2$$

$$\Delta'\equiv 2\lambda\mu\nu$$

$$\Theta\equiv 2(gh-af)\lambda+2(hf-bg)\mu+2(fg-ch)\nu$$

$$\Theta'\equiv -a\lambda^2-b\mu^2-c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu.$$

* Salmon—Conic Sections, § 372.

These conditions are equivalent to the two equations—

$$3\Delta\Theta'=\Theta^2 \quad\text{and}\quad 3\Delta'\Theta=\Theta'^2.$$

The first equation is one of order two in λ, μ, ν, and consequently represents tangentially a conic touched by the line

$$\lambda x+\mu y+\nu z=0,$$

i.e., the six inflexional tangents touch the conic—

$$3\Delta\Theta'=\Theta^2 \qquad \dots \quad (4)$$

The second equation represents a curve of the fourth class, which is also touched by the six inflexional tangents.

The point-equation of the conic (4) is obtained in the form—

$$4(a\mathrm{A}^2x^2+b\mathrm{B}^2y^2+c\mathrm{C}^2z^2+2f\mathrm{BC}yz+2g\mathrm{CA}zx+2h\mathrm{AB}xy)$$

$$=\Delta(\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+2\mathrm{F}yz+2\mathrm{G}zx+2\mathrm{H}xy)$$

$$+4\Delta(afyz+bgzx+chxy),$$

CONIC VI: *The six inflexions of a trinodal quartic lie on a conic.*

Brill, by applying the theory of quadratic forms, determined a cubic passing through the inflexions. Then by the application of certain identities and quadratic transformation obtained the actual expression for the conic which passes through the six points of inflexion.

We give here a direct proof of the theorem, which has been given by Richmond and Stuart * and is considered to be simpler and less complicated.

* Richmond and Stuart—*Proc. Lond. Math. Soc.*, Vol. II (1) (1903), p. 130.

Consider the conics (1) and (2) of the last theorem, and suppose they have a three-point contact at (x', y', z') and let

$$L \equiv px+qy+rz=0$$

be the line joining this point to their fourth intersection so that

$$px'+qy'+rz'=0.$$

The tangent to (2) at (x', y', z') being—

$$T \equiv x(ax'+hy'+gz')+y(hx'+by'+fz')+z(gx'+fy'+cz')=0$$

we must have $\quad S \equiv R+kTL=0.$*

Equating the co-efficients of x^2, y^2, z^2 on the right-hand side to zero, we obtain—

$$k=p(ax'+hy'+gz')/a=q(hx'+by'+fz')/b=r(gx'+fy'+cz')/c$$

whence

$$\frac{ax'}{ax'+hy'+gz'}+\frac{by'}{hx'+by'+fz'}+\frac{cz'}{gx'+fy'+cz'}=0.$$

Therefore, the locus of the point of osculation (x', y', z') is the cubic—

$$\Theta=a x(hx+by+fz)(gx+fy+cz)+\cdots+\cdots=0$$

Since the point (x', y', z') lies on the conic (2), it also lies on the locus

$$R(ghx+hfy+fgz)=\Theta,$$

which may be written as

$$2Ayz(hy+gz)+2Bzx(fz+hx)+2Cxy(gx+fy)$$
$$+(4abc-4fgh-\Delta)xyz=0.$$

If now we replace x, y, z by $1/x$, $1/y$, $1/z$, the resulting equation will give a curve passing through the points where

* Salmon—Conic Sections, § 251.

the line $\lambda x+\mu y+\nu z=0$ osculates the quartic, *i.e.*, the points of inflexion.

Thus, the inflexions are found to lie on the cubic—

$$\Psi\equiv 2\mathrm{A}x^2(gy+hz)+2\mathrm{B}y^2(hz+fx)+2\mathrm{C}z^2(fx+gy)$$
$$+(4abc-4fgh-\Delta)xyz=0.$$

This cubic meets the quartic in twelve points, namely, twice at each vertex of the triangle of reference and the six inflexions. But the conic—

$$\Phi\equiv ghyz+hfzx+fgxy=0$$

is found to touch the cubic at the vertices, and consequently it passes through six of the twelve intersections, and hence by Cayley's theorem,* the other six intersections, namely, the inflexions, must lie on a conic.

The actual expression for the conic is obtained from the identity—

$$\mathrm{KU}+\mathrm{M}\Psi\equiv\Phi\{2\mathrm{A}(g\mathrm{G}+h\mathrm{H})x^2+\cdots+\cdots$$
$$-(4f\mathrm{F}^2+2gh\Delta+3af\Delta)yz-\cdots-\cdots\}$$

where

$$\mathrm{K}\equiv(afgh\mathrm{A}+bfgh\mathrm{B}+cfgh\mathrm{C}+2bcgh\mathrm{F}+2cahf\mathrm{G}+2abfg\mathrm{H})$$

and $\quad \mathrm{M}\equiv f(g\mathrm{G}+h\mathrm{H})x+g(h\mathrm{H}+f\mathrm{F})y+h(f\mathrm{F}+g\mathrm{G})z.$

Thus the six inflexions lie on the conic—

$$2\mathrm{A}(g\mathrm{G}+h\mathrm{H})x^2+\cdots+\cdots$$
$$-(4f\mathrm{F}^2+2gh\Delta+3af\Delta)yz-\cdots-\cdots=0,$$

i.e., on the conic—

$$2(a\mathrm{A}^2x^2+\ldots+\ldots+2f\mathrm{BC}yz+\ldots+\ldots)$$
$$-2\Delta(\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2-\mathrm{F}yz-\mathrm{G}zx-\mathrm{H}xy)$$
$$+\Delta(afyz+bgzx+chxy)=0.$$

* Theory of Plane Curves, Vol. I, § 30.

CONIC VII: *The six points where the nodal tangents meet the quartic again lie on a conic.**

Let A_1, A_2, B_1, B_2 and C_1, C_2 be the points, where the tangents at A, B, C respectively meet the quartic again.

Let $\lambda x+\mu y+\nu z=0$ represent the line C_1C_2.

The lines joining the node C(0, 0, 1) to the four points where C_1C_2 intersects the quartic are given by—

$$(ay^2+bx^2+2hxy)(\lambda x+\mu y)^2-2xy(fx+gy)\nu(\lambda x+\mu y) \\ +c\nu^2x^2y^2=0.$$

The nodal tangents $ay^2+bx^2+2hxy=0$ are one pair of these lines, and if

$$Px^2+2Qxy+Ry^2=0$$

represents the other pair, by identifying the equation with

$$(ay^2+bx^2+2hxy)(Px^2+2Qxy+Ry^2)=0$$

we obtain,

$$\lambda : \mu : \nu=gbc : fca : -2(fF+gG)$$

and $\quad P : Q : R=b^2cg^2 : fg(abc+4fgh-2af^2-2bg^2) : a^2cf^2.$

Thus the line C_1C_2 is represented by the equation

$$\frac{bc}{f}x+\frac{ca}{g}y=\frac{2(fF+gG)}{fg}z.$$

Similarly, the lines A_1A_2, B_1B_2 are represented by

$$\frac{ca}{g}y+\frac{ab}{h}z=\frac{2(gG+hH)}{gh}x$$

and $$\frac{ab}{h}z+\frac{bc}{f}x=\frac{2(hH+fF)}{fh}y.$$

* The proof given here is taken from Prof. Hilton's Plane Algebraic Curves, Chap. XVII, p. 276. For a second proof, see Basset's *Cubics and Quartics*, § 194.

But from the identity—

$$(bz^2+cy^2+2fyz)(az^2+cx^2+2gzx)-cU$$
$$\equiv z^2\{abz^2+2(2fg-ch)xy+2bgzx+2afyz\}$$

it follows at once that the four points A_1, A_2 and B_1, B_2 lie on the conic—

$$abz^2+2(2fg-ch)xy+2bgzx+2afyz=0 \quad \ldots \quad (1)$$

Consequently, the said four points also lie on the conic—

$$k\{abz^2+2(2fg-ch)xy+2bgzx+2afyz\}$$
$$=\left\{-\frac{2(gG+hH)}{gh}x+\frac{ca}{g}y+\frac{ab}{h}z\right\}$$
$$\left\{\frac{bc}{f}x-\frac{2(hH+fF)}{hf}+\frac{ab}{h}z\right\}.$$

If now we can show that this conic also passes through C_1 and C_2 for some value of k, the proposition is proved. In fact, in that case, for some value of k, the above equation will be symmetrical, and by comparing the co-efficients of x^2, y^2, z^2, the required value of k is found to be—

$$\{2h(fF+gG)+abfg\}/fgh^2$$

and for this value of k, the equation of the conic becomes—

$$2bc(gG+hH)x^2+\cdots+\cdots$$
$$=a(abcf+2bcgh+4af^2-8f^2gh)yz+\cdots+\cdots$$

It is to be noticed that the conic (1) passes through the nodes A and B. Thus we obtain the theorem:

A conic can be described through any two nodes and the four points at which the tangents at these nodes meet the quartic again.

CONIC VIII: *The six points of contact of the tangents drawn from the nodes to the quartic lie on a conic.*

The method of proof is exactly the same as in the preceding case.

Let the points of contact be denoted by α_1, α_2; β_1, β_2 and γ_1, γ_2 respectively.

The equations of the tangents are given in § 153.

Proceeding as before, the lines $\alpha_1\alpha_2$, $\beta_1\beta_2$, $\gamma_1\gamma_2$, are found to be given by—

$$\mathrm{B}y/g+\mathrm{C}z/h=(g\mathrm{G}+h\mathrm{H})x/gh$$

$$\mathrm{C}z/h+\mathrm{A}x/f=(h\mathrm{H}+f\mathrm{F})y/hf,$$

and $$\mathrm{A}x/f+\mathrm{B}y/g=(f\mathrm{F}+g\mathrm{G})z/fg.$$

From the identity—

$$(\mathrm{B}y^2-2\mathrm{F}yz+\mathrm{C}z^2)(\mathrm{A}x^2-2\mathrm{G}zx+\mathrm{C}z^2)-\Delta\mathrm{U}$$
$$\equiv(\mathrm{C}z^2-\mathrm{F}yz-\mathrm{G}zx+\mathrm{H}xy)^2$$

the points $\alpha_1, \alpha_2, \beta_1, \beta_2$ are found to lie on the conic—

$$\mathrm{C}z^2-\mathrm{F}yz-\mathrm{G}zx+\mathrm{H}xy=0.$$

Consequently, they must also lie on the conic—

$$k(\mathrm{C}z^2-\mathrm{F}yz-\mathrm{G}zx+\mathrm{H}xy)$$
$$=\{(\mathrm{B}y/g+\mathrm{C}z/h-(g\mathrm{G}+h\mathrm{H})x/gh\}$$
$$\{\mathrm{A}x/f+\mathrm{C}z/h+(h\mathrm{H}+f\mathrm{F})y/hf\}.$$

This conic also passes through γ_1 and γ_2, if we take—

$$k=(fh\mathrm{F}+gh\mathrm{G}+fg\mathrm{C})/fgh^2,$$

and the equation then reduces to—

$$a\mathrm{A}^2x^2+b\mathrm{B}^2y^2+c\mathrm{C}^2z^2+2f\mathrm{BC}yz+2g\mathrm{CA}zx+2h\mathrm{AB}xy$$
$$=\Delta(\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2-\mathrm{F}yz-\mathrm{G}zx-\mathrm{H}xy)$$

157. A geometrical proof of the theorem that the six inflexions lie on a conic is interesting.*

The geometrical treatment is effected by the combination of the methods of projection and cyclic inversion.

If two nodes are projected into the two circular points at infinity, the system of conics circumscribing the nodal triangle is transformed into a system of circles passing through a fixed point, namely, the projection of the third node. Since there are six inflexional tangents which correspond to six conics circumscribing the nodal triangle and osculating the *adjoint* conic, we have six circles passing through a fixed point and osculating the conic which is the projection of the *adjoint* conic. The centres of all these circles lie on a conic.†

Now, invert cyclically with regard to the fixed point. Taking one of the circles, the end of the diameter through the fixed point lies on a conic C (similar to the conic of centres). It is easy to see that the first negative pedal of the inverse of C is the inverse of the point of osculation, but the same negative pedal is the polar reciprocal of the inverse. Hence the polar reciprocal of the inverse of the inverse of C is the inverse of the point of osculation. Thus the point of osculation lies on the inverse of the polar reciprocal of C, which is a conic, and its inverse is a trinodal quartic having the fixed point and the two circular points as nodes. This trinodal quartic then passes through all the six points of osculation.

Now, going back to the original system, this quartic corresponds to a quartic co-trinodal with the original quartic, and the problem is solved.

* This proof was communicated by Mr. A. C. Bose of the University of Calcutta.

† Malet's Theorem—See Casey's Analytical Geometry (1893), p. 471.

Ex. 1. Show that the bitangential conic, the conic through the inflexions and the conic touched by the inflexional tangents pass through the same four points.

[If S_1, S_2 and S_3 are the conics respectively, prove that

$$S_2 - 2S_3 \equiv 3\Delta S_1]$$

Ex. 2. Show that the points of contact of tangents drawn from A and B are collinear, if the tangents from C are harmonic conjugates with respect to CA and CB.

[The conic $Cz^2 - Fyz - Gzx + Hxy = 0$ reduces to a pair of lines, if $H = 0$ and the tangents from C are then $Ax^2 + By^2 = 0$].

Ex. 3. The tangents at C are harmonic conjugates *w.r.t.* CA and CB. Shew that the points of contact of tangents drawn from A lie on a line through B, and that the remaining intersections of the curve with the tangents at A lie on a line through the same point.

[We have $h = 0$. The equation of $\alpha_1\alpha_2$ becomes $Gx - Cz = 0$.]

Ex. 4. The nodal tangents meet the quartic again respectively in A_1, A_2; B_1, B_2; C_1, C_2. Shew that the intersections of $A_1 A_2$ and BC, B_1B_2 and CA, C_1C_2 and AB are collinear.

[The line is $bcghx + cahfy + abfgz = 0.$]

Ex. 5. Find the condition that the eight points of contact of the bitangents lie on two lines [The bitangential conic degenerates.]

158. Tricuspidal Quartics:

A unicursal quartic may have all its double points as cusps and it is called a tricuspidal quartic. The results established for a trinodal quartic are applicable, with some modifications, to tricuspidal quartics. A tricuspidal quartic can be projected into a Cardioid or a three-cusped hypocycloid, and it is the reciprocal polar of a nodal cubic. Consequently, its properties can be derived from those of these latter three curves.

If in the equation of the trinodal quartic, we assume that the tangents at each double point are coincident, we must have—

$$f = \pm\sqrt{bc}, \quad g = \pm\sqrt{ca} \quad \text{and} \quad h = \pm\sqrt{ab}$$

Hence the equation of the tricuspidal quartic takes the form—

$$a/x^2+b/y^2+c/z^2 \pm 2\sqrt{bc}/yz \pm 2\sqrt{ca}/zx \pm 2\sqrt{ab}/xy=0$$

We may take, without loss of generality, all the signs negative. Then putting $\sqrt{a}\,x$, $\sqrt{b}\,y$, $\sqrt{c}\,z$ for x, y, z respectively, the above equation reduces to the form—

$$1/x^2+1/y^2+1/z^2-2/yz-2/zx-2/xy=0$$

or, $$y^2z^2+z^2x^2+x^2y^2-2xyz(x+y+z)=0 \quad \ldots \quad (1)$$

which may be written in the form—

$$x^{-\frac{1}{2}}+y^{-\frac{1}{2}}+z^{-\frac{1}{2}}=0$$

The cuspidal tangents are $x=y$, $y=z$ and $z=x$, which evidently meet in the point $x=y=z$.

Applying the quadric inversion, *i.e.* putting $1/X$, $1/Y$, $1/Z$ for x, y, z, we obtain the equation of the adjoint conic

$$X^2+Y^2+Z^2-2YZ-2ZX-2XY=0$$

which can be written as—

$$(X+Y-Z)^2=4XY.$$

Now putting $X+Y-Z=2\mu X$, we obtain $4\mu^2X^2=4XY$

$$i.e. \quad \mu^2X=Y$$

Again, we have $2\mu X(X+Y-Z)=4XY$

$\therefore\ \mu(X+Y-Z)=2Y$, and consequently $(1-\mu)^2X=Z$

Thus $X : Y : Z=1 : \mu^2 : (1-\mu)^2$

and consequently,

$$x : y : z=1 : \frac{1}{\mu^2} : \frac{1}{1-\mu^2}$$

$$=\mu^2(1-\mu^2) : (1-\mu)^2 : \mu^2$$

i. e., the co-ordinates are expressed in terms of a parameter μ.

Writing the equation (1) in the form

$$(yz+zx+xy)^2=4xyz(x+y+z)$$

it is at once evident that the line $x+y+z=0$ is the only bitangent which is also real. The points of contact are its intersections with the conic $yz+zx+xy=0$. It will thus be readily seen that the points of contact are real, when two of the cusps are imaginary, and imaginary, when all the cusps are real.

159. THEOREM : *Any tricuspidal quartic may be projected into a three-cusped hypocycloid or into a Cardioid.*

Since any triangle can be projected into an equilateral triangle such that any given point may coincide with any given point, we may project the tricuspidal quartic, such that the cuspidal triangle projects into a *real* equilateral triangle whose centre of gravity coincides with the point of concurrence of the cuspidal tangents. The projection is then a three-cusped hypocycloid, or again, if we project the points of contact of the bitangent into the circular points at infinity, the projection is a three-cusped hypocycloid, which is a tricuspidal quartic, the line at infinity touching it at the circular points.

If, however, we project two of the cusps of a tricuspidal quartic into the circular points, the projection is a limaçon with a finite cusp, *i.e.*, a Cardioid, which is the inverse of a parabola *w.r.t.* the focus.

Ex. 1. Show that the theorem that the three inflexions of a nodal cubic are collinear reciprocates into the theorem—The cuspidal tangents of a tricuspidal quartic are concurrent.

Ex. 2. Three conics are drawn through the cusps touching the curve elsewhere. Show that the remaining intersections of the conics are collinear.

[By inverting *w.r.t.* the focus of a parabola, the theorem that the circle circumscribing the triangle formed by three tangents passes through the focus, we obtain a Cardioid and three circles through the finite cusp, touching it, with their other common points collinear, which, when projected into a tricuspidal quartic, proves the theorem.]

160. QUARTICS WITH A NODE AND TWO CUSPS:

When the nodes B and C of the trinodal quartic (§ 153) reduce to cusps, we must have $ca=g^2$ and $ab=h^2$, and the equation of a quartic having a node at A and two cusps at B and C may be written as—

$$a/x^2+b/y^2+c/z^2+2fyz\pm2\sqrt{ca}/zx\pm2\sqrt{ab}/xy=0.$$

Now replacing x, y, z respectively by

$$\sqrt{a}\,x,\quad \sqrt{b}\,y,\quad \sqrt{c}\,z,$$

we may write the equation, taking the negative sign in the ambiguity, in the form—

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{2f}{yz\sqrt{bc}}-\frac{2}{zx}-\frac{2}{xy}=0$$

i.e. $$y^2z^2+z^2x^2+x^2y^2+\frac{2f}{\sqrt{bc}}x^2yz-2xy^2z-2xyz^2=0$$

which may again be written as—

$$y^2z^2+z^2x^2+x^2y^2=2xyz(kx+y+z) \qquad \dots \quad (1)$$

In order to express the equation in the parametric form, consider its intersections with a conic through A, B, C, touching the cuspidal tangent at B(say); for instance—

the conic $\qquad y(z-x)=tzx$

meets the quartic in a variable point whose co-ordinates can be expressed in terms of t.

Writing the equation (1) in the form

$$(yz+zx+xy)^2=2xyz\{(k+1)x+2y+2z\}$$

it is at once inferred that the line

$$(k+1)x+2y+2z=0$$

is a bitangent, the points of contact lying on the conic

$$yz+zx+xy=0.$$

Eliminating x between this equation and the equation of the bitangent, it is found that the lines joining the points of contact to the node are given by

$$2y^2+2z^2-(k-3)yz=0.$$

The line joining the inflexions may be obtained by considering the intersections of the curve and its Hessian, and if x be eliminated between their equations, we obtain the equation of the lines joining the inflexions to the node A in the form—

$$9(k+1)(y^2+z^2)=2(k^2+k+16)yz.$$

The reciprocal of a quartic with one node and two cusps is a similar quartic, and consequently each of its properties can be duplicated by reciprocation. When the cusps are projected into the circular points, the curve becomes the inverse of a conic with respect to a focus, *i.e.*, a limaçon.

Ex. 1. Show that the bitangent, the line through the cusps and the inflexional line are concurrent.

Ex. 2. If a quartic has cusps at B and C and a node at A, show that the points of contact of the tangents drawn from the cusps and those of the bitangent lie on a conic through the cusps.

Ex 3. Prove that the cusps, the inflexions and the points where the nodal tangents meet the quartic again lie on a conic.

Ex. 4. Prove that the inflexions are unreal or real, according as k does or does not lie between 1 and 7.

Ex. 5. Prove that the locus of the intersection of the tangents at the extremities of a nodal chord is a nodal cubic through the cusps.

[This can be easily established by projecting the cusps into the circular points at infinity. The projected curve is a limaçon, the corresponding theorem becomes—The locus of intersection of the tangents at the extremities of a nodal chord is a nodal circular cubic. Or, prove it directly from the equation of the curve.]

161. THEOREM:

In a quartic with two cusps and a node, the nodal tangents, together with the lines joining the node to the intersection of the cuspidal tangents and to that of the bitangent with the line through the cusps, form a harmonic pencil.

Let the equation of the quartic be taken in the form—

$$y^2z^2+z^2x^2+x^2y^2=2xyz(kx+y+z) \qquad \cdots \quad (1)$$

Let P be the intersection of the cuspidal tangents and Q the point where the bitangent meets the line BC through the cusps.

The tangents at the cusps B and C are respectively

$$z-x=0 \quad \text{and} \quad x-y=0$$

so that the equation of AP is $\qquad y=z \qquad \ldots \quad (2)$

The bitangent is the line—

$$(k+1)x+2y+2z=0$$

so that the line AQ is given by—

$$y+z=0 \qquad \ldots \quad (3)$$

The nodal tangents are given by—

$$y^2+z^2-2kyz=0$$

Writing $k=\cosh\theta$, these tangents become—

$$\left.\begin{aligned} y-ze^{\theta}&=0 \\ y-ze^{-\theta}&=0 \end{aligned}\right\} \qquad \ldots \quad (4)$$

which shows that the four lines (2), (3) and (4) form a harmonic pencil.

If the two cusps are projected into the circular points we obtain the theorem: *In a limaçon, the nodal tangents the line joining the node with the triple focus and the line drawn through the node parallel to the bitangent form a harmonic pencil.*

162. QUARTICS WITH COMPLEX SINGULARITIES EQUIVALENT TO THREE DOUBLE POINTS :

I. *Quartics with a tacnode and another double point* :

We shall show that a tacnodal quartic can also be obtained from a conic by the application of the quadric transformation already explained.

The equation of a quartic having a tacnode at C with $y=0$ as the tacnodal tangent may be written as—

$$(yz-mx^2)(yz-m'x^2)$$
$$=axy^2z+by^3z+cx^3y+dx^2y^2+exy^3+fy^4$$

If we suppose that the point B is a node, we must have $f=b=e=0$, and the equation reduces to—

$$(yz-mx^2)(yz-m'x^2)=axy^2z+cx^3y+dx^2y^2 \quad \ldots \quad (1)$$

Writing this equation in the form—

$$(yz)^2+b'x^2(yz)+c'(xy)(yz)+d'(x^2)^2$$
$$+e'(x^2)(xy)+f'(xy)^2=0$$

it is seen that it is a quadratic function of (xy), (x^2) and (yz). Hence, replacing x, y, z respectively by xy, x^2, yz in the general equation of a conic, we obtain the equation of a quartic with a tacnode and a node.

Thus, if we write—

$$x' : y' : z'=xy : x^2 : yz$$

we obtain reciprocally—

$$x : y : z=x'y' : x'^2 : y'z'$$

so that by applying a special quadric transformation,* we may study the properties of this quartic.

* Theory of Plane Curves, Vol. I, § 218 (1).

Similarly, when the node is a cusp or the tacnode becomes a node-cusp, we may obtain similar transformations, so that the theory applies to the case of unicursal quartics with two distinct double points, one of them a node or cusp and the other a tacnode or a node-cusp.

In particular cases, the above equation can be reduced to a simpler form, for instance, the sides z and x may be taken as the harmonic conjugates of the nodal tangents at B, we must have then $a=0$ and the equation (1) becomes—

$$(yz-mx^2)(yz-m'x^2)=x^2y(cx+dy)$$

or, the same may be reduced, by changing constants, to the form—

$$(yz+x^2)^2=(1-e)x^2\{x^2+pxy+qy^2\} \quad \ldots \quad (2)$$

If B is a cusp, we must have $d=0$, *i.e.*, $q=0$.

Ex. 1. Obtain the equations of the bitangents of the quartic—

$$(yz+x^2)^2=(1-e)x^2(x^2+pxy+qy^2)$$

[The bitangents are $2z=(1\pm\sqrt{e})(px+qy)$, the points of contact lying on the conic

$$2yz+2x^2=(1-e)(2x^2+pxy+qy^2)]$$

Ex. 2. Prove that the points of contact of tangents drawn from A lie on a conic which touches the quartic at the tacnode.

[The first polar (polar cubic) of A consists of the side BC and the required conic.]

Ex. 3. Prove that the ratio of the radii of curvature of the two branches of the quartic (any curve in general) is unaltered by projection.

Ex. 4. Show that the equation

$$y^2z^2+2zyu_2+u_4=0$$

represents a quartic with a tacnode at C. Obtain its equation when B is a node or a cusp.

II. *Quartics with an oscnode* :

The equation of a quartic with an oscnode at C is of the form—

$$(yz-mx^2)^2+cxy(yz-mx^2)+dy^3z+ex^2y^2+fxy^3+gy^4=0$$

which is evidently a quadratic function of $yz-mx^2$, xy and y^2, and from the transformation—

$$x' : y' : z'=xy : y^2 : yz-mx^2$$

we obtain reciprocally

$$x : y : z=x'y' : y'^2 : y'z'-mx'^2$$

Thus the theory also extends to this case as well,* *i.e.*, we may obtain its properties from those of a conic by quadric inversion.

For particular values of the constants, the equation can be made to represent a quartic with a tacnode-cusp.

Ex. Obtain the equation of a quartic with a superlinear branch of order three in the form $y^3z=(3ay^2-x^2)^2$. [See Theory of Plane Curves, Vol. I, § 266.]

III. *Quartics with a triple point* :

A quartic with a triple point is also a unicursal curve, but the foregoing method is not applicable. In this case, in order to express the co-ordinates of any point in terms of a parameter, we consider a line drawn through the triple point, which intersects the curve in only one other point.

Thus, if C is the triple point, the equation of the quartic is $zu_3=u_4$, where u_3 and u_4 are polynomials of order three and four respectively in x, y.

Now, consider the line $y=tx$. The intersections are given by $z\Theta_3=x\Theta_4$, where Θ_3 and Θ_4 denote cubic and quartic functions of t. Hence $x : y : z=\Theta_3 : t\Theta_3 : \Theta_4$.

* Theory of Plane Curves, Vol. I, § 218.

By a suitable choice of the constants, the equation of the cubic can be reduced to simpler forms. Suppose that the real tangents at the triple point are x, y, $(x+y)$, so that $u_3 \equiv xy(x+y)$. Replacing z by $z+px+qy$ and then determining p, q such that the co-efficients of x^3y and xy^3 are zero, the equation can be written in the form—

$$xyz(x+y)=ax^4+2hx^2y^2+by^4 \qquad \dots \quad (3)$$

If any two points of the quartic are projected into the circular points and then the curve inverted with respect to the triple point, we obtain a unicursal cubic. Hence from the properties of a unicursal cubic, we may deduce the properties of a quartic with a triple point. (See Ex. 5, § 124.)

Thus from the theorem that the tangentials of three collinear points of a cubic are collinear, we obtain the theorem:

The three conics osculating a quartic at the triple point and passing through two other points on the quartic meet the quartic again in points lying on a conic through the three points.

Ex. 1. Show that the quartic $x^3y=z^4$ has an undulation and that it is the reciprocal of a curve of similar nature.

Ex. 2. Prove that

$$\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s}=0$$

represents a quartic with a triple point, having p, q, r, s as bitangents, where p, q, r, s are such that the lines $p=s$, $q=s$ and $r=s$ are concurrent.

Ex. 3. If the equation of a quartic with a triple point can be reduced to the form—

$$xyz(x+y)=ax^4+2hx^2y^2+by^4$$

point out the relation with the sides of the triangle of reference.

Ex. 4. If two tangents at the triple point at C coincide with the line $x=0$, the equation can be put into the form —

$$x^2yz=ax^4+y^3(2x+y).$$

Show that the inflexions lie on the conic $3yz+2zx=2ax$

163. UNICURSAL QUARTICS WITH FLECNODES AND BIFLECNODES:

If C is a flecnode on the trinodal quartic (1) of § 153, the co-efficients of z and z^2 should have a common linear factor,

i.e., $\qquad xy(fx+gy)$ and $bx^2+2hxy+ay^2$

must have a common linear factor, which requires that

$$2fgh=af^2+bg^2$$

Similarly, when B is a flecnode, we must have—

$$2fgh=ch^2+af^2.$$

Both these conditions are found to be satisfied, if we put

$$a=4h^2q,\ b=p/(p+q)^2,\ c=f^2p \quad \text{and} \quad g=2fh(p+q),$$

so that the equation of a quartic with flecnodes at B and C can be written as—

$$4h^2qy^2z^2+pz^2x^2/(p+q)^2+4f^2px^2y^2$$
$$+2xyz\{fx+2fh(p+q)y+hz\}=0 \qquad \dots \quad (1)$$

If A is to be a flecnode, we must have $p=q$, and the equation reduces to a perfect square. Hence, we obtain the theorem that *a quartic cannot have more than two flecnodes.*

Assuming suitable values for the constants in the equation (1), it may be reduced to the simpler form—

$$2(f-1)y^2z^2+z^2x^2+x^2y^2+2fxyz(x+y+z)=0$$

In the case of a biflecnode, $xy(fx+gy)$ should contain

$$bx^2+2hxy+ay^2$$

as a factor, and this requires that $f=g=0$.

The equation of a quartic with three biflecnodes is—

$$a/x^2+b/y^2+c/z^2=0 \qquad \dots \quad (2)$$

In order that the quartic may be real, it is necessary that the sign of one of the constants should be different from those of the other two. If now we write $-c$ for c, it follows that when a, b, c are all positive, the nodal tangents at A and B are all real, while those at C are imaginary. Hence, of the three biflecnodes, one is an acnode and the other two crunodes.

Putting $\sqrt{a}\,x$, $\sqrt{b}\,y$, $\sqrt{c}\,z$ for x, y, z respectively, the equation of such a quartic can be written as—

$$x^{-2}+y^{-2}=z^{-2} \qquad \ldots \quad (3)$$

The tangents at the crunodes A and B are $y\pm z$ and $z\pm x$ respectively, which form a quadrilateral with diagonals intersecting at $(0, 0, 1)$, *i.e.*, the acnode.

The co-ordinates of any point on the curve (3) may be expressed in the form

$$x : y : z=1/(1-t^2) : 1/2t : 1/(1+t^2)$$

or, $$=\sec\theta : \operatorname{cosec}\theta : 1, \quad \text{where} \quad t=tan\tfrac{1}{2}\theta.$$

Again, the quartic may be real, if two biflecnodes are imaginary and the third one real.

Consider the equation—

$$\lambda\beta^2\gamma^2+\mu\gamma^2\alpha^2+\nu\alpha^2\beta^2=0$$

where α, β, γ are any real or imaginary lines forming a triangle.

If now we put $\alpha=x$, $\beta=y+ikz$, $\gamma=y-ikz$, the equation can be written in the form—

$$\lambda(y^2+k^2z^2)^2-x^2(py^2-2kqyz-pk^2z^2)=0$$

This is then the equation of a quartic having a real biflecnode at A and two imaginary ones at the points where x intersects the lines $y+ikz$ and $y-ikz$.

If now we change the constants, *i.e.*, put

$$k=1, \text{ and } \lambda : \mu : \nu = 1 : i : -i,$$

the equation of a quartic having a real biflecnode at A and two imaginary ones at the points where the lines $y+iz$, $y-iz$ meet the side x can be put into the form—

$$(y^2+z^2)^2=2x^2yz$$

If, again, the line x moves off to infinity, and y, z are two rectangular axes through the real node A, we obtain the equation in the form—

$$(y^2+z^2)^2=2a^2yz$$

i.e.,
$$r^2=a^2\sin 2\theta,$$

which is the lemniscate of Bernoulli.

Ex. 1. Obtain the equation of a quartic having a pair of flecnodes at B and C and a cusp at A in the form—

$$x(\lambda/x+\mu/y+\nu/z)^2=4\lambda(\lambda/x+\nu/z)$$

Ex. 2. Find the bitangents of the quartic in Ex. 1.

Ex. 3. Shew that every quartic having three biflecnodes may be projected into a lemniscate of Bernoulli.

Ex. 4. A quartic has flecnodes at B and C and a node at A. Show that the points of contact of the tangents drawn from A, B, C lie on a conic passing through B and C.

Ex. 5. The points where the tangents at A, B, C meet the quartic again lie on a second conic through B and C.

Ex. 6. Show that a unicursal curve cannot have—

(*i*) a flecnode and two cusps (*ii*) a flecnode and a biflecnode

(*iii*) a biflecnode and a cusp.

CHAPTER IX

BINODAL AND BICIRCULAR QUARTICS

164. QUARTICS WITH DEFICIENCY ONE OR TWO:

Quartics with deficiency one or two have been studied in connection with the theory of bitangents. The case when the quartic is of deficiency two has received little attention and no attempt is made for a complete discussion here. Quartics of deficiency one, *i.e.*, possessing two nodes (or cusps), specially the case where these are the circular points at infinity, have been exhaustively studied under the name of *bicircular* quartics.* All the projective properties of these curves may therefore be derived from those of binodal quartics, although some may be obtained directly. Quartics with two cusps at the circular points have been studied under the name of *Cartesians*,† whose properties may again be derived from those of bicuspidal quartics. We prefer to insert at the end of this volume (Appendix II) some of the results as obtained by Dr. Casey.

It will be noticed that bicircular quartics belong to the species IV, VII or VIII and the Cartesians belong to the species VI, IX or X.

The case when one circular point is a cusp and the other a node on a quartic, when the curve is certainly imaginary, has accordingly been little noticed. It may be pointed out that for a binodal quartic a theory of the inscription of polygons, analogous to Poncelet's theory in regard to conics, has been developed by Steiner.‡ [See the references given under article 14.]

* For discussion of the complete theory, see Casey's paper—Proc. of the Royal Irish Academy, Vol. 10, p. 44 (1869).

† Cayley—This curve was first studied by Des Cartes. Liouville—Vol. 15, p. 354.

‡ Steiner—*Geometrische Lehrsätze, Crelle*, Bd. 32, p. 186 (1846).

165. THE EQUATION OF A BICIRCULAR QUARTIC:

The general equation of a quartic curve is—

$$u_0+u_1+u_2+u_3+u_4=0$$

where $u_4=0$ represents four right lines drawn through the origin, parallel to the asymptotes of the curve.

If the curve passes through the circular points I and J, the circular lines $y\pm ix=0$ are parallel to the asymptotes whose points of contact are at I or J. If, again, I and J are nodes on the curve, the nodal tangents at each of I and J reduce to two coincident tangents, which are consequently two coincident asymptotes of the curve. Hence, the four asymptotes reduce to a pair of coincident lines, parallel to

$$y\pm ix=0$$

Thus, $$u_4\equiv(y+ix)^2=(x^2+y^2)^2$$

Again, each of the lines $y\pm ix=0$ must meet the curve in two coincident points at I or J.

Let $$u_3\equiv ax^3+3bx^2y+3cxy^2+dy^3$$

Substituting $y=\pm ix$ in the equation of the curve, we find

$$0.\,x^4+(a\pm 3ib-3c\mp id)x^3+const.\,x^2+const.\,x+u_0=0.$$

Now, since each of the lines $y\pm ix=0$ meets the curve in two points at I or J, two roots of this equation should be infinite, *i.e.*, the co-efficients of the two highest powers of x should vanish.

The co-efficient of x^4 is zero; the co-efficient of x^3 will be zero, if

$$a\pm 3ib-3c\mp id=0$$

whence $$3b=d \text{ and } 3c=a.$$

$$\therefore\quad u_3\equiv(x^2+y^2)(ax+dy).$$

Thus the equation of a quartic having two nodes at the circular points I and J may be written as—

$$k(x^2+y^2)^2+(x^2+y^2)(ax+dy)+u_2+u_1+u_0=0;$$

i.e., in the form—

$$(x^2+y^2)^2+2(lx+my)(x^2+y^2)$$
$$+ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots \quad (1)$$

This equation contains eight independent constants, as it should be; for the general equation of a quartic contains 14 disposable constants; that I and J are nodes on the curve is equivalent to six conditions, and consequently reduces the number of constants by six.

166. OTHER FORMS OF THE EQUATION:

The above equation can be written in the symbolic form—

$$r^4+2u_1r^2+v_2+v_1+v_0=0.$$

Adding and subtracting u_1^2 to the equation, we obtain—

$$(r^4+2u_1r^2+u_1^2)=u_1^2-v_2-v_1-v_0,$$

i.e.,
$$(r^2+u_1)^2=u_1^2-v_2-v_1-v_0.$$

Now, r^2+u_1 on the left evidently represents a circle and the right-hand side is the equation of a conic. Hence the equation of a bicircular quartic can be written in the form $C^2=S$, or, $C^2=I.S$, where C is a circle, S is any conic, and I is the line at infinity.

The second form of the equation shows that the conic S and the line at infinity each touches the quartic at the points where the quartic is met by the circle C. The circle meets the quartic twice at each of the two circular points at

infinity and in four finite points, and at these four points the conic S touches the quartic. The line at infinity meets the circle at the circular points, which are two nodes on the curve.

Conversely, if the equation of a quartic can be brought to the form $C^2 = S$, where C is a circle and S any conic, the curve is a bicircular quartic.

Again, we may write—

$$(C+k)^2 = S + 2kC + k^2$$

i.e., $C'^2 = S'$, where C' represents a circle and S' represents the conic $S + 2kC + k^2$.

Now, since there is only one constant k involved in this reduction, the equation can be reduced to the above form in a singly infinite number of ways. The circles $C+k$ evidently have the same centre for all values of k, *i.e.*, it represents a system of concentric circles.

Ex. Show that the equation of a bicircular quartic can be written in the form—

$$au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv = 0$$

where u, v, w represent three circles.

167. Generation of Binodal and Bicircular Quartics:

We have seen that the equation of a quartic can be expressed as a quadratic function of U, V, W, where U, V, W represent three conics. Thus we may regard its equation reduced to the form $UW = V^2$, without loss of generality. For a non-singular quartic, these conics cannot have a common point, for, the common point becomes a double point on the quartic. In the case of binodal quartics, however, U, V, W may be taken as three conics passing each through the two nodes. Thus a binodal quartic may be regarded as the envelope of—

$$\lambda^2 U + 2\lambda V + W = 0,$$

where U, V, W are three conics through the two nodes.

If the two nodes are at the circular points at infinity, *i.e.*, in the case of bicircular quartics, the conics U, V, W are all circles. Hence we may regard a bicircular quartic as the envelope of the variable circle—

$$\lambda^2 U + 2\lambda V + W = 0$$

where U, V, W represent three circles.

Again, the enveloping conic $\lambda^2 U + 2\lambda V + W = 0$ touches the quartic at the four points given by—

$$\lambda U + V = 0, \qquad \lambda V + W = 0.$$

Now, when U, V, W are all circles, besides the two circular points, the two circles $\lambda U + V$ and $\lambda V + W$ intersect in two other finite points, which are the points of contact of the enveloping circle with the bicircular quartic.

Therefore, each of the variable circles $\lambda^2 U + 2\lambda V + W = 0$ has double contact with the bicircular quartic, the chord of contact being the radical axis of the two circles—

$$\lambda U + V = 0, \qquad \lambda V + W = 0.$$

Now, if L, M, N be the linear parts in the equations of U, V, W respectively, the radical axis is—

$$\lambda(L - M) + (M - N) = 0,$$

which evidently passes through the fixed point $L = M = N$, *i.e.*, the radical centre of the three circles U, V, W.

Thus the enveloping circle has double contact with the bicircular quartic, the chord of contact always passing through a fixed point.

Ex. Show that at the point of contact of a tangent drawn from the radical centre of U, V, W, a circle can be drawn having four-point contact with the curve, and that these points are equidistant from the radical centre.

168. It is proved in Treatises on Conics * that the Jacobian of three circles consists of the line at infinity and another circle which cuts them all orthogonally and that the Jacobian of three circles included in the form—

$$lU + mV + nW = 0$$

is the same as that of the circles U, V, W. Hence, the circles included in the above form, and in particular, in the form—

$$\lambda^2 U + 2\lambda V + W = 0$$

have a common Jacobian, *i.e.*, a common orthogonal circle. Thus the enveloping circles always cut a fixed circle orthogonally.

Again, if (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are the co-ordinates of the centres of the three circles U, V, W, the co-ordinates of the centre of the variable circle are given by—

$$x : y : z = \lambda^2 x_1 + 2\lambda x_2 + x_3 \ : \lambda^2 y_1 + 2\lambda y_2 + y_3$$
$$: \lambda^2 z_1 + 2\lambda z_2 + z_3$$

Eliminating λ between these equations, we obtain the locus of the centre x, y, z as a conic. Hence, from what has been said in the preceding article, it follows that—

The bicircular quartic $UW = V^2$ may be regarded as the envelope of a variable circle whose centre moves along a fixed conic and which cuts a fixed circle orthogonally.

When U, V, W are conics through the two nodes of the binodal quartic $UW = V^2$, it may be regarded as the envelope of the variable conic—

$$\lambda^2 U + 2\lambda V + W$$

passing through the fixed points; the variable conics have a common Jacobian conic J, and the locus of the pole of the line joining the fixed points *w.r.t.* the conics of the system is a fixed conic F.

* Salmon—Conics § 388, *Ex.* 3.

Ex. 1. If the conics J and F touch, show that the point of contact is an additional node of the binodal quartic.

Ex. 2. If J and F touch twice, show that the quartic reduces to two conics. [The quartic has four nodes, etc.]

Ex. 3. If the conic F passes through a node of the binodal quartic, the node reduces to a cusp, and when F passes through both the nodes, the quartic becomes a bicuspidal quartic.

Ex. 4. If the locus of centres F is a circle for a bicircular quartic, the curve becomes a Cartesian.

Ex. 5. When F touches the line joining the nodes, show that the quartic reduces to a cubic and the line; and deduce that if the conic of centres of a bicircular quartic is a parabola, the quartic degenerates into a circular cubic together with the line at infinity.

169. THE CONVERSE THEOREM :

The envelope of a variable circle whose centre moves along a fixed conic and which cuts a fixed circle orthogonally is a bicircular quartic.

Let
$$x^2+y^2+2fx+2gy+c=0 \qquad \dots \quad (1)$$
be a fixed circle,

and
$$x^2/a^2+y^2/b^2=1 \qquad \dots \quad (2)$$
be a fixed conic.

The co-ordinates of any point on the conic may be taken as $(a\cos\theta, b\sin\theta)$, and the equation of a variable circle of radius r, with centre $(a\cos\theta, b\sin\theta)$ may be written as—

$$(x-a\cos\theta)^2+(y-b\sin\theta)^2=r^2 \qquad \dots \quad (3)$$

Now, the condition that this circle cuts (1) orthogonally gives—

$$(a\cos\theta+f)^2+(b\sin\theta+g)^2=(f^2+g^2-c)+r^2$$

i.e., $r^2=c+2af\cos\theta+2bg\sin\theta+a^2\cos^2\theta+b^2\sin^2\theta \; \dots \quad (4)$

Eliminating r^2 between (3) and (4), we obtain the equation of a variable circle which cuts (1) orthogonally and whose centre moves along (2).

The equation of the variable circle thus becomes—

$$x^2+y^2-2ax\cos\theta-2by\sin\theta=c+2af\cos\theta+2bg\sin\theta$$

or, $$2a(x+f)\cos\theta+2b(y+g)\sin\theta=x^2+y^2-c \quad \cdots \quad (5)$$

In order to determine the envelope of (5), as θ varies, we differentiate it with respect to θ, and obtain—

$$-2a(x+f)\sin\theta+2b(y+g)\cos\theta=0 \quad \ldots \quad (6)$$

Squaring (5) and (6), and adding, we obtain the envelope of the variable circle (5) in the form—

$$4a^2(x+f)^2+4b^2(y+g)^2=(x^2+y^2-c)^2 \quad \ldots \quad (7)$$

which represents a bicircular quartic.

170. Centre of inversion:

Transferring the origin to the centre $(-f,-g)$ of the fixed circle, the equation (7) reduces to—

$$4a^2x^2+4b^2y^2=\{x^2+y^2-2fx-2gy+\delta^2\}^2,$$

where δ is the radius of the fixed circle,

i.e., $$4(a^2x^2+b^2y^2)=(r^2-u_1+\delta^2)^2 \quad \ldots \quad (8)$$

If now the quartic be inverted, referred to the new origin, with respect to the fixed circle, it is inverted into itself. For, the inverse of (8) is—

$$4\frac{\delta^4}{r^4}(a^2x^2+b^2y^2)=\left\{\frac{\delta^4}{r^4}r^2-\frac{\delta^2}{r^2}u_1+\delta^2\right\}^2$$

or, $$4(a^2x^2+b^2y^2)=(\delta^2-u_1+r^2)^2$$

i.e., the curve is self-inverse with regard to the fixed circle.

It is on this account that the centre $(-f,-g)$ of the fixed circle is called the *centre of inversion*, and the fixed circle is called the *circle of inversion*. The variable circle is called the *generating circle*. The fixed conic is called the *focal* or 'deferent' *conic*; for, as will be shown later on, it passes through four foci of the quartic.

171. THEOREM:

There are four circles of inversion and four corresponding focal conics associated with any bicircular quartic.

Taking the centre of the fixed circle as origin and the axes of co-ordinates parallel to the principal axes of the conic, the equation of the quartic can be written as—

$$a'x^2+b'y^2+2f'x+2g'y+c'=(x^2+y^2)^2 \quad \dots \quad (9)$$

or, introducing a parameter λ,

$$(a'-2\lambda)x^2+(b'-2\lambda)y^2+2f'x+2g'y+c'+\lambda^2$$
$$=(x^2+y^2-\lambda)^2 \quad \dots \quad (10)$$

The conic—

$$(a'-2\lambda)x^2+(b'-2\lambda)y^2+2f'x+2g'y+c'+\lambda^2=0$$

evidently touches the quartic at four points lying on the circle $x^2+y^2=\lambda$.

Now, this equation can be written in the form—

$$(a'-2\lambda)\left\{x^2+\frac{2f'}{a'-2\lambda}x+\frac{f'^2}{(a'-2\lambda)^2}\right\}$$
$$+(b'-2\lambda)\left\{y^2+\frac{2g'y}{b'-2\lambda}+\frac{g'^2}{(b'-2\lambda)^2}\right\}$$
$$+\left\{c'+\lambda^2-\frac{f'^2}{a'-2\lambda}-\frac{g'^2}{b'-2\lambda}\right\}=(x^2+y^2-\lambda)^2,$$

i.e., $$(a'-2\lambda)\left\{x+\frac{f'}{a'-2\lambda}\right\}^2+(b'-2\lambda)\left\{y+\frac{g'}{b'-2\lambda}\right\}^2$$
$$+\left\{c'+\lambda^2-\frac{f'^2}{a'-2\lambda}-\frac{g'^2}{b'-2\lambda}\right\}=(x^2+y^2-\lambda)^2 \quad \dots \quad (11)$$

Thus, if this is to be of the same form as (7) of § 169, we must have—

$$c'+\lambda^2-\frac{f'^2}{a'-2\lambda}-\frac{g'^2}{b'-2\lambda}=0$$

which is a biquadratic in λ and gives four values of λ. Hence there are *four* different ways of reducing the equation of a bicircular quartic to the form (7).

Corresponding to each value of λ, we obtain a circle of inversion and a corresponding focal conic. By comparing the equation (11) with (7) of § 169, we obtain—

$$4a^2 = a' - 2\lambda;\ 4b^2 = b' - 2\lambda;\ f = \frac{f'}{a'-2\lambda},\ g = \frac{g'}{b'-2\lambda},\ c = \lambda$$

Therefore the equation of the focal conic is—

$$4x^2/(a'-2\lambda) + 4y^2/(b'-2\lambda) = 1 \qquad \ldots \quad (12)$$

which represents a system of confocal conics for different values of λ.

The equation of the circle of inversion becomes—

$$x^2 + y^2 + \frac{2f'x}{a'-2\lambda} + \frac{2g'y}{b'-2\lambda} + \lambda = 0 \qquad \ldots \quad (13)$$

which represents a circle whose centre is the point

$$\left(-\frac{f'}{a'-2\lambda},\ -\frac{g'}{b'-2\lambda}\right)$$

Eliminating λ between the three equations—

$$x = \frac{-f'}{a'-2\lambda},\quad y = \frac{-g'}{b'-2\lambda} \quad \text{and} \quad c' + \lambda^2 - \frac{f'^2}{a'-2\lambda} - \frac{g'^2}{b'-2\lambda} = 0$$

the four centres of inversion are given as the intersections of the two rectangular hyperbolas —

$$(a'-b')xy + f'y - g'x = 0$$

and $4f'g'(x^2 - y^2) + 4(g'^2 - f'^2)xy$

$$+ g'(4c' - a'^2)x + f'(b'^2 - 4c')y - (a'-b')f'g' = 0.$$

Therefore, by a known theorem in conics, it follows that the four centres of inversion are such that each is the orthocentre of the triangle formed by the other three.

Thus a bicircular quartic may be regarded as *the envelope of a variable circle which cuts any of the four fixed circles orthogonally and whose centre moves along any of the four corresponding confocal conics (focal conics). The centres of these four fixed circles are such that each is the orthocentre of the triangle formed by the other three.*

172. FOCAL CONICS:

The four points of intersection of a circle of inversion with the corresponding focal conic are the foci of the bicircular quartic.

Let P, Q, R, S be the four points of intersection of the fixed circle with the fixed conic (§ 169).

Consider the variable circle with centre at the point P, cutting the circle of inversion orthogonally. If r be the radius of the variable circle and r' that of the fixed circle, since the two circles are orthogonal, we must have—

$$r'^2 = r'^2 + r^2$$

whence $r=0$, *i.e.*, the variable circle is of zero radius, or a point-circle. But each generating circle has double contact with the quartic (§ 167). Hence, at P we have a point circle which has double contact with the curve, *i.e.*, the point P is a focus. Similarly, the other three intersections Q, R, S are also foci of the bicircular quartic.

Since there are four circles of inversion and four corresponding focal conics associated with any bicircular quartic, we obtain Dr. Hart's theorem, namely, *there are sixteen foci of a bicircular quartic, which lie on four circles, four on each circle.*

173. A THEOREM ON BINODAL QUARTICS:

The above theorem can be deduced from a more general theorem on binodal quartics, namely,—

*The anharmonic ratios of the two pencils of four tangents, which can be drawn to a binodal quartic from the two nodes, are equal.**

The equation of a binodal quartic, having nodes at the vertices B and C of the triangle of reference, is—

$$y^2z^2 + 2xyz(my+nz) + x^2(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)=0.$$

* *Cf.* Cayley—"Memoir on Polyzomal Curves"—Coll. Papers, Vol. 6, p. 529 (1868).

The pairs of tangents at B and C are respectively given by—

$$z^2+2mzx+bx^2=0 \quad \text{and} \quad y^2+2nxy+cx^2=0.$$

Now choose y and z to be the harmonic conjugates of x with respect to the nodal tangents at C and B respectively. This requires that $m=0$ and $n=0$, and the nodal tangents are—

$$z^2+bx^2=0 \quad \text{and} \quad y^2+cx^2=0.$$

The equation of the quartic can then be written as—

$$f\equiv y^2z^2+x^2(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)=0 \quad \dots \quad (1)$$

The equation of the four tangents drawn from the point B may be obtained by first writing down the equation of the first polar of B and then eliminating y between this and the equation of the curve. But the result of elimination of y between the first polar $df/dy=0$ and $f=0$ is the same as the condition for a double root of the equation $f=0$, regarded as an equation in y. Thus, the equation of the tangents is—

$$(z^2+bx^2)(ax^2+2gzx+cz^2)=x^2(fz+hx)^2$$

i.e.,
$$(ab-h^2)x^4+(2bg-2fh)x^3z+(a+bc-f^2)x^2z^2$$
$$+2gxz^3+cz^4=0 \quad \dots \quad (2)$$

The six anharmonic ratios of this pencil are given by

$$\frac{I^3 *}{27J^2}=\frac{(\sigma^2-\sigma+1)^3}{(\sigma+1)^2\,(\sigma-2)^2\,(\sigma-\frac{1}{2})^2} \quad \dots \quad (3)$$

where I and J are the invariants of (2).

* Burnside and Panton—Theory of Equations, Vol. I, Ex. 16, pp. 148-150.

$$\text{But}\quad \mathrm{I}=c(ab-h^2)-4.\ \frac{2(bg-fh)}{4}\ .\ \frac{2g}{4}+3.\ \frac{(a+bc-f^2)^2}{36}$$

$$=abc-ch^2-g(bg-fh)+\tfrac{1}{12}(a+bc-f^2)^2$$

$$=abc-ch^2-bg^2+fgh+\tfrac{1}{12}(a+bc-f^2)^2$$

$$\text{and}\quad \mathrm{J}=(ab-h^2)\frac{(a+bc-f^2)}{6}.\,c+2.\frac{2(bg-fh)}{4}.\ \frac{(a+bc-f^2)}{6}.\ \frac{2g}{4}$$

$$-(ab-h^2)\frac{4g^2}{16}-c.\ \frac{4(bg-fh)^2}{16}-\frac{(a+bc-f^2)^3}{216}$$

$$i.e.,\quad 6\mathrm{J}=(abc-ch^2)(a+bc-f^2)+\tfrac{1}{2}(bg^2-fgh)(a+bc-f^2)$$

$$-\tfrac{3}{2}g^2(ab-h^2)-\tfrac{3}{2}c(bg-fh)^2-\tfrac{1}{36}(a+bc-f^2)^3.$$

$$=(abc-ch^2-bg^2-\tfrac{1}{2}fgh)(a+bc-f^2)-\tfrac{3}{2}f^2(ch^2+bg^2)$$

$$+3bcfgh+\tfrac{3}{2}g^2h^2-\tfrac{1}{36}(a+bc-f^2)^3.$$

Thus we see that I and J are symmetrical between b and c, as well as g and h; and consequently the equation (3) remains unchanged when we interchange b and c, as also g and h. But, by this interchange we obtain only the equation of the four tangents drawn from C. Hence the anharmonic ratios of the two pencils are equal.

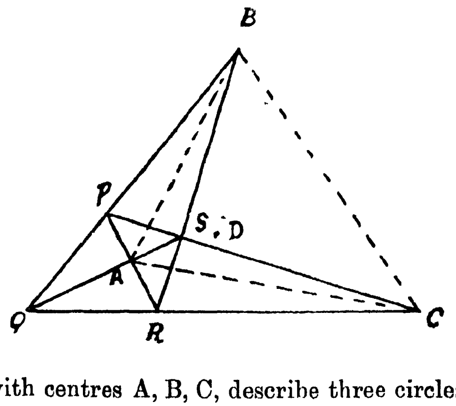
Since the two pencils are homographic, a conic can be described through the points of intersection of the corresponding rays of the two pencils, which also passes through the nodes. Again, since there are four orders in which the legs of the second pencil can be taken without altering the anharmonic ratio, it follows that the sixteen points of intersection of the first pencil with the second lie on four conics, each passing through the two nodes.

In the case of a bicircular quartic, the nodes are at the circular points and the points of intersection of the tangents become the foci of the curve, whence the truth of Dr. Hart's theorem follows.

174. FOUR MUTUALLY ORTHOGONAL CIRCLES OF INVERSION:

Let P, Q, R, S be four concyclic foci, *i.e.*, the four points in which a circle of inversion intersects the corresponding focal conic. Let PR, QS; PQ, RS; and PS, QR respectively intersect in the three points A, B, C. The triangle ABC is self-conjugate with respect to all conics through PQRS. Hence it is self-conjugate with respect to the circle of inversion, and consequently the centre of the circle is the orthocentre D of the triangle ABC, and the four points A, B, C, D are such that each is the orthocentre of the triangle formed by the other three. Thus we may state the following theorem:

The centres of three circles for which the bicircular quartic is self-inverse are the vertices of the common self-conjugate triangle of the fourth circle and its focal conic.



Now, with centres A, B, C, describe three circles U, V, W such that the orthocentre D is the radical centre of the three circles. Then the circle of inversion intersects orthogonally each of U, V, W, and consequently also the circle—

$$lU+mV+nW=0$$

But the envelope of $lU+mV+nW=0$ is the bicircular quartic—

$$\lambda U^2+\mu V^2+\nu W^2=0$$

provided $$l^2/\lambda+m^2/\mu+n^2/\nu=0 \qquad \ldots \quad (1)$$

The co-ordinates of the centre of the variable circle are found to be proportional to l, m, n. Hence, from (1) the centre moves along the conic—

$$x^2/\lambda+y^2/\mu+z^2/\nu=0 \tag{2}$$

which is a conic *w.r.t.* which the triangle ABC is self-conjugate, *i.e.*, it is one of the conics through P, Q, R, S.

Thus the bicircular quartic may be generated as the envelope of the variable circle—

$$l\mathrm{U}+m\mathrm{V}+n\mathrm{W}=0$$

which cuts the circle of inversion I, as well as each of the circles U, V, W orthogonally, and whose centre moves along the focal conic (2).

If follows, therefore, that the bicircular quartic can also be generated as the envelope of a variable circle which cuts each of the four mutually orthogonal circles I, U, V, W orthogonally, and whose centre moves along a corresponding focal conic.

175. The four centres of inversion:

We shall now show that the points A, B, C, D are four centres of inversion of the bicircular quartic, that is to say, the quartic is self-inverse with respect to each of the circles with centres A, B, C and D.

Since each of the circles U, V, W intersects I orthogonally, the radii r_1, r_2, r_3 of U, V, W respectively are the tangents drawn to I from their centres A, B, C.

Take A to be the origin, AB the axis of x, and a line through A perpendicular to AB as the axis of y; and put

$$\mathrm{BC}=a, \quad \mathrm{CA}=b, \quad \mathrm{AB}=c.$$

Then the co-ordinates of B and C are respectively—

$$(c, 0) \quad \text{and} \quad (b\cos \mathrm{A}, b\sin \mathrm{A})$$

$\therefore$ The equations of U, V, W are respectively—

$$U \equiv x^2+y^2-r_1^2=0$$

$$V \equiv (x-c)^2+y^2-r_2^2=x^2+y^2-2cx+c^2-r_2^2=0$$

$$W \equiv (x-b\cos A)^2+(y-b\sin A)^2-r_3^2$$

$$=x^2+y^2-2b(x\cos A+y\sin A)+(b^2-r_3^2)=0$$

But $a^2=r_2^2+r_3^2$; $b^2=r_3^2+r_1^2$; $c^2=r_1^2+r_2^2$

$$\therefore \quad U \equiv x^2+y^2-r_1^2=0 \quad V \equiv x^2+y^2-2cx+r_1^2=0$$

$$W \equiv x^2+y^2-2b\ (x\cos A+y\sin A)+r_1^2=0.$$

Now, the equation of the quartic is—

$$\lambda U^2+\mu V^2+\nu W^2=0.$$

Let us invert this with respect to the circle U.

Now, if the inverses of U, V, W *w.r.t.* the circle U are respectively denoted by U′, V′, W′, then, we must have—

$$r^2U'=-r_1^2U; \quad r^2V'=r_1^2V; \quad r^2W'=r_1^2W$$

and the inverse of—

$$\lambda U^2+\mu V^2+\nu W^2=0$$

is the same curve, which proves the proposition.

Thus A is a centre of inversion. Similarly B and C are also centres of inversion.

Ex. 1. Prove that the four circles of inversion are mutually orthogonal.

Ex. 2. Show that the four centres of inversion form four triangles having a common nine-points circle.

Ex. 3. Prove that the radical axis of two circles is the line joining the centres of the remaining two.

Ex. 4. Show that if the four real foci of a bicircular quartic are concyclic, the curve consists of two ovals, and three of the four circles of inversion are real.

176. THEOREM :

If a circle of inversion touches its corresponding focal conic, the point of contact is a node on the quartic ; and when it osculates the focal conic, the point of contact is a cusp.

Let P (x', y') be the point where the focal conic touches the circle of inversion given in § 169. Then the equation of the bicircular quartic referred to O, the centre of the focal conic, as origin is—

$$4a^2(x+f)^2+4b^2(y+g)^2=(x^2+y^2-c)^2 \qquad \dots \quad (1)$$

Since the circle and the conic have a common tangent at the point P (x', y'), we have the following conditions satisfied—

$$x'^2+y'^2+2fx'+2gy'+c=0 \qquad \dots \quad (2)$$

$$x'^2/a^2+y'^2/b^2=1 \qquad \dots \quad (3)$$

$$\frac{a^2(x'+f)}{x'}=\frac{b^2(y'+g)}{y'}=-(fx'+gy'+c) \qquad \dots \quad (4)$$

Now, transferring the origin to the point (x', y') the equation (1) becomes—

$$4a^2(x+x'+f)^2+4b^2(y+y'+g)^2$$
$$=\{(x+x')^2+(y+y')^2-c\}^2 \qquad \dots \quad (5)$$

The constant term in this equation is—

$$4a^2(x'+f)^2+4b^2(y'+g)^2-(x'^2+y'^2-c)^2$$

which vanishes by virtue of the relations (2), (3) and (4)

The linear term is—

$$x\{8a^2(x'+f)-4x'(x'^2+y'^2-c)\}$$
$$+y\{8b^2(y'+g)-4y'(x'^2+y'^2-c)\}$$

and by virtue of the said relations, the co-efficients of both x and y are found to be zero.

Hence, the point P is a double point on the curve,

The quadratic term is found to be—

$$x^2\{4(x'^2-a^2)+2(x'^2+y'^2-c\}$$
$$+y^2\{4(y'^2-b^2)+2(x'^2+y'^2-c)\}+8x'y'xy$$

which reduces to

$$4x^2\{x'^2-a^2-(fx'+gy'+c)\}$$
$$+4y^2\{y'^2-b^2-(fx'+gy'+c)\}+8x'y'xy$$

Now, from relations (4) we easily obtain—

$$-(fx'+gy'+c)=\frac{(x'+f)}{\frac{x'}{a^2}}=\frac{(y'+g)}{\frac{y'}{b^2}}$$

$$=\frac{\sqrt{(x'+f)^2+(y'+g)^2}}{\sqrt{x'^2/a^4+y'^2/b^4}}=p\delta,$$

where p is the length of the perpendicular drawn from O on the tangent, and δ is the radius of the circle of inversion.

Thus the equation of the tangents is reduced to

$$x^2\{x'^2+p\delta-a^2\}+y^2\{y'^2+p\delta-b^2\}+2x'y'xy=0$$

Hence, the point P will be a node, a cusp or a conjugate point, according as

$$x'^2y'^2\gtreqless(x'^2+p\delta-a^2)(y'^2+p\delta-b^2)$$

i.e., according as $\quad p\delta(a^2+b^2-x'^2-y'^2)-p^2\delta^2\gtreqless 0$

i.e., according as $\quad p\delta-a^2b^2/p^2\lesseqgtr 0$

since in the conic, we have $a^2b^2/p^2=a^2+b^2-(x'^2+y'^2)$

Now, if ρ is the diameter conjugate to OP, we have

$$\rho p=ab, \qquad i.e. \qquad \rho^2=a^2b^2/p^2$$

Hence, the point P is a node, a cusp or a conjugate point, according as

$$p\delta-\rho^2 \lesseqqgtr 0, \qquad i.e., \text{ according as } \qquad \rho^2 \gtreqqless p\delta.$$

When $\rho^2=p\delta$, δ becomes the radius of curvature at P, *i.e.*, the circle of inversion osculates the conic at the point P, which is a cusp.

It is to be noticed that the point of contact P must be real, otherwise the quartic would have three imaginary double points. Hence, two real foci coincide at P.

Thus, *when a bicircular quartic possesses a real node, it arises from the union of two real single foci, and when there is a real cusp, it is formed by the union of three real single foci.*

Ex. 1. If the circle of inversion has double contact with the focal conic, the bicircular quartic reduces to two circles.

[Each point of contact is a double point. The curve then possessing four double points degenerates into two conics through the circular points.]

Ex. 2. Prove that the circles of curvature of a bicircular quartic at its intersections with a circle of inversion have four-point contact.

177. THE BITANGENTS OF A BINODAL QUARTIC:

We have already seen that the generating conic—

$$\lambda^2U+2\lambda V+W=0$$

of the quartic $UW=V^2$ breaks up into two right lines for six different values of λ, and these right lines are bitangents of the quartic. In the case of a binodal quartic, the conics—

$$U, \ V, \ W \quad \text{and} \quad \lambda^2U+2\lambda V+W$$

all pass through the two nodes; and consequently, when this latter represents right lines, it denotes two lines passing one through each of the nodes, or else, the line through

the nodes and another line. In the first case, the lines are not proper bitangents, but ordinary tangents through the nodes. In the second case, one is a proper bitangent, and the other is the nodal line. Thus we cbtain—

A binodal quartic has only two proper bitangents.

For, when U, V, W have two common points, V and W will be of the forms—

$$kU+LM \quad \text{and} \quad k'U+LN;$$

and $$\lambda^2 U+2\lambda V+W$$

becomes $$(\lambda^2+2\lambda k+k')U+L(2\lambda M+N)$$

Hence, this will have L for a factor, if $\lambda^2+2\lambda k+k'=0$

The two values of λ given by this equation will give us *two* bitangents of the form $2\lambda M+N=0$. The other four values of λ, which reduces the equation to the product of two linear factors, give the tangents drawn through the nodes.

Ex. 1. Show how the equation of a binodal quartic with nodes at A and B can be reduced to the form—

$$(x^2+pz^2)(y^2+qz^2)+2kxyz^2+2(fx+gy+hz)z^3=0$$

Ex. 2. Prove that the points of contact of the tangents drawn from C to the quartic of *Ex.* 1 lie on the conic—

$$qx^2+py^2+2(pq+2h)z^2+gyz+fzx+kxy=0$$

[The first polar of C degenerates.]

Ex. 3. Prove that the points of contact of any two pairs of bitangents of a binodal quartic lie on a conic.

Ex. 4. Prove that the sixteen intersections of tangents drawn to a binodal quartic from the nodes lie by fours on four conics touching the nodal tangents.

Ex. 5. All binodal quartics with two given nodes passing through seven other fixed points pass through an eighth fixed point.

[Project the nodes into circular points and inverting *w.r t.* one fixed point, obtain a pencil of circular cubics through six points.]

178. PROPER BITANGENTS OF BICIRCULAR QUARTICS :

When U, V, W are three circles, the curve has two nodes at the circular points.

In this case, if $\lambda^2U+2\lambda V+W$ is to represent two right lines, one of them will be the line at infinity and the other will touch the curve in two points and will therefore be a proper bitangent.

But $\lambda^2U+2\lambda V+W=0$ is now of the form—

$$(a\lambda^2+2b\lambda+c)(x^2+y^2)+\text{linear terms}=0$$

Hence, we must have $a\lambda^2+2b\lambda+c=0$, which gives two values of λ. Corresponding to each value, we obtain a proper bitangent whose equation may be written in the form—

$$2\lambda M+N=0$$

and this evidently passes through the point $M=N=0$, *i.e.*, the radical centre of U, V, W, which is a centre of inversion.

This also follows from the fact that the enveloping circle has double contact with the curve, the chord of contact passing through the radical centre.

Thus through a centre of inversion there pass two proper bitangents of the bicircular quartic; and since there are four centres of inversion, *a bicircular quartic has eight proper bitangents.*

We may obtain the same results from other considerations. If the variable circle becomes a right line, its centre passes to infinity, and must consequently be the point at infinity on one of the two asymptotes of the fixed conic. The two bitangents are therefore the perpendiculars drawn from the centre of the Jacobian on these asymptotes.

179. The same thing also follows from the equation—

$$4a^2(x+f)^2+4b^2(y+g)^2=(x^2+y^2-c)^2$$

Writing this equation in the form—

$$\{2a(x+f)+2ib(y+g)\}\{2a(x+f)-2ib(y+g)\}$$
$$=(x^2+y^2-c)^2$$

the factors on the left represent the two bitangents through the point $(-f,-g)$.

These are real or imaginary, according as the focal conic

$$x^2/a^2+y^2/b^2=1$$

is a hyperbola or an ellipse.

Thus, through a centre of inversion there pass two bitangents of the curve, and consequently there are eight bitangents.

The generating circle (§ 169)

$$2a(x+f)\cos\theta+2b(y+g)\sin\theta=x^2+y^2-c \quad \dots \quad (1)$$

touches the quartic, the chord of contact being—

$$2b(y+g)\cos\theta-2a(x+f)\sin\theta=0 \quad \dots \quad (2)$$

which evidently passes through the fixed point $(-f, -g)$ *i.e.*, the centre of the fixed circle.

Thus we obtain a more general theorem:

If a circle has double contact with a bicircular quartic, the chord of contact passes through one of four fixed points.

Ex. 1. Prove that the bitangents through the centre of a circle of inversion are perpendicular to the asymptotes of the corresponding focal conic.

Ex. 2. Show that the directrices corresponding to four concyclic foci are concurrent in the centre of the circle.

Ex. 3. Show how the equation of a bicircular quartic can be reduced to the form—

$$(x^2+y^2)^2+ax^2+2hxy+by^2+2gx+2fy+c=0$$

180. Cyclic Points:

If the line (2) is a tangent to (1), it will also be a tangent to the bicircular quartic at a point where the circle (1) has a third order contact with the curve, *i.e.*, the point of contact is a *cyclic* point on the curve.

Now, the line (2) touches (1), when the perpendicular drawn from the centre on the line is equal to the radius of the circle, *i.e.*, the condition of contact is a function of the parameter and gives certain finite values of θ, and therefore the curve has a finite number of cyclic points, which are the points of contact of the tangents drawn from a centre of inversion. Thus we obtain the theorem:

The points of contact of tangents drawn from a centre of inversion to a bicircular quartic are the cyclic points on the curve.

Again, since the generating circle (1) intersects the circle of inversion orthogonally, whose centre lies on the line (2), it follows that, when the line (2) touches the circle (1), the point of contact lies on the circle of inversion and this point is a cyclic point on the quartic. Hence the four finite points in which the circle of inversion cuts the bicircular quartic are cyclic points on the latter.

Since there are four circles of inversion, these circles intersect the quartic in sixteen finite points, which are cyclic points on the curve. The same thing also follows from the fact that, from each centre of inversion four ordinary tangents can be drawn to the curve, whose points of contact are cyclic points. We have seen that these points also lie on the circles of inversion. Therefore the lengths of the four tangents drawn from a centre of inversion are equal. Thus we obtain the theorem:

Each bicircular quartic has sixteen cyclic points which lie on four circles, four on each circle.

Ex. The normals at the points where a bicircular quartic meets a circle of inversion touch the corresponding focal conic.

181. Singular Foci :

Consider the bicircular quartic represented by—

$$4a^2(x+f)^2+4b^2(y+g)^2=(x^2+y^2-c)^2$$

The singular foci of the curve are the real points of intersection of the nodal tangents at the two circular points I and J.*

The circular line $y=ix+k$ intersects the curve in points whose abscissae are determined by the equation—

$$4a^2(x+f)^2+4b^2(ix+k+g)^2=\{k(2ix+k)-c\}^2$$

If, therefore, the line is to be a nodal tangent, three roots of this equation should be infinite; and consequently, the co-efficients of x^4, x^3 and x^2 must vanish.

It is evident that for all values of k, the co-efficients of x^4 and x^3 are zero. The co-efficient of x^2 will be zero,

if, $$4a^2-4b^2+4k^2=0$$

i.e., if, $$a^2-b^2=-k^2$$

whence $$k=\pm i\sqrt{a^2-b^2}$$

Therefore, the equations of the nodal tangents at a circular point are—

$$y=i(x\pm\sqrt{a^2-b^2})$$

Similarly, the nodal tangents at the other circular point are—

$$y=-i(x\pm\sqrt{a^2-b^2})$$

Hence, the singular (double) foci are the real points of intersection of these tangents, namely, the points—

$$y=0, \qquad x=\pm\sqrt{a^2-b^2}=\pm ae$$

which are the common foci of the system of focal conics. Similarly, the other two imaginary singular

* Theory of Plane Curves, Vol. I, § 169.

foci can be determined, and they are found to be the imaginary points

$$(0, \pm\sqrt{b^2-a^2}).$$

Thus we may state the following theorem :

The singular foci of a bicircular quartic are the foci of the focal conic.

182. THE INVERSE OF A BICIRCULAR QUARTIC :

The inverse of a bicircular quartic is another bicircular quartic ; but when the origin of inversion lies on the curve, the inverse is a circular cubic.

The general equation of a bicircular quartic can be written as—

$$u_0r^4+u_1r^2+v_2+v_1+v_0=0$$

The inverse of this with respect to the origin is—

$$v_0r^4+k^2v_1r^2+k^4v_2+k^6u_1+k^8u_0=0$$

which is also a bicircular quartic.

If, however, the origin lies on the curve, the constant term v_0 is absent, and the inverse reduces to—

$$v_1r^2+k^2v_2+k^4u_1+k^6u_0=0$$

which is evidently a circular cubic.

Ex. 1. The bisectors of the angle between any pair of bitangents of a bicircular quartic are parallel and perpendicular to the line joining the real singular foci.

Ex. 2. A two-circuited bicircular quartic consists of two ovals, one inside the other. Prove that the focal conic through the real foci is a hyperbola and the other focal conics are ellipses.

Ex. 3. If in *Ex.* 2 the ovals are external to each other, the focal conic through the real foci is an ellipse and the other focal conics are hyperbolas.

Ex. 4. Show that the four centres of inversion lie on a rectangular hyperbola whose asymptotes are parallel and perpendicular to the line joining the real singular foci and which passes through the mid-point of this line.

183. INVERSES OF THE FOCI :

Since the foci of a curve are inverted into the foci of its inverse, if we invert a bicircular quartic with respect to a circle of inversion, the sixteen foci will be inverted into the sixteen foci of the inverse curve, and although the curve is its own inverse, each individual focus will not be inverted into itself. We shall now investigate the relation between these foci and their inverse points.

Let α_1, α_2, α_3, α_4 be the foci on the circle with centre A. Similarly, let $(\beta_1, \beta_2, \beta_3, \beta_4)$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and $(\delta_1, \delta_2, \delta_3, \delta_4)$ be the groups of foci on the circles with rentsce B, C, and D respectively.

Invert the curve with respect to the circle A. Then the foci $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are inverted into themselves. The inverse of β_1, a focus on B, will be a focus of the inverse curve, *i.e.*, a focus of the curve itself. Therefore, the inverse of β_1 must be one of the remaining 11 foci, *i.e.*, the line $A\beta_1$ must pass through one of these remaining foci. Three different cases are to be considered :—

The inverse of β_1 may be—

(*i*) one of the three points $\beta_2, \beta_3, \beta_4$;

(*ii*) one of the points of the γ group ;

(*iii*) one of the points of the δ group.

(1) When the inverse of β_1 is one of the points β_2, β_3, β_4, the circle B is inverted into itself, and consequently, the γ group is inverted into the δ group, or they are the inverses of themselves.

(2) When β_1 is inverted into one of the foci of the γ group, then the circle B is inverted into the circle C, and consequently, the points A, B, C are collinear. In this case, the δ group cannot be inverted into any of the three groups α, β, or γ. For, the α group is its own inverse and β group inverts into the γ group. Consequently, the δ group is inverted into itself.

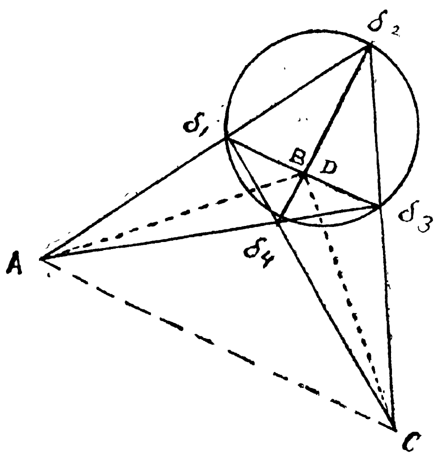
(3) In this case the β group is inverted into the δ group, while the γ group is inverted into itself.

Now, since the bicircular quartic has four circles of inversion, any property which holds for any two must also hold for the remaining two. The relation between the four circles must be symmetrical. Hence, the only symmetrical relation which may exist between them is that each of the groups is inverted into itself.

This will be true also for the other three centres of inversion. Hence, the four circles of inversion, when inverted with respect to any one of them, are inverted into themselves (§ 175).

184. THE FOUR CENTRES OF INVERSION:

When A is the centre of inversion, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are inverted each into itself. But δ_1 cannot be inverted into itself, and since the circle D is inverted into itself, the inverse of δ_1 will be one of the three foci $\delta_2, \delta_3, \delta_4$ (say δ_2). Then A, δ_1, δ_2 are in one right line. Consequently A, δ_3, δ_4 are in one right line.



Thus, A is a diagonal point of the quadrangle $\delta_1\delta_2\delta_3\delta_4$. Similarly, if B is the centre of inversion, $\delta_1, \delta_2, \delta_3, \delta_4$ are

inverted into themselves, and consequently B is collinear with $\delta_1\delta_3$ and $\delta_2\delta_4$. Hence B is a diagonal point of the same quadrangle. Similarly, the centre C is the third diagonal point of $\delta_1\delta_2\delta_3\delta_4$. Also, D is the centre of the circle $\delta_1\delta_2\delta_3\delta_4$.

Hence ABC is self-conjugate with respect to the circle whose centre is D, and consequently D is the orthocentre of the triangle ABC.

Similarly, it can be shown that any of the four points A, B, C, D, will be the orthocentre of the triangle formed by the other three, as was otherwise shown in § 174.

Ex. Show how to find the singular focus of a bicircular quartic when the circles of inversion and any point on the curve are given.

185. CIRCLES OF INVERSION MUTUALLY ORTHOGONAL:

From what has been said above it follows that the circles of inversion cut each other orthogonally. For, considering the circles A and D, δ_1 and δ_2 are inverse points with respect to the circle A.

$$\therefore\ A\delta_1 . A\delta_2 = r^2$$

where r is the radius of the circle A.

But the product $A\delta_1 . A\delta_2$ is equal to the square of the length of the tangent drawn from A to the circle D.

$$\therefore\ A\delta_1 . A\delta_2 = r^2 = \text{square of the tangent from A to D.}$$

Thus the point of contact of the tangent drawn from A is a point on the circle A, *i.e.*, the two circles A and D cut orthogonally.

Similarly, the same relation will be found to exist between any two of them.

Hence the circles cut each other orthogonally, as was otherwise shown in article 174.

186. Inverse of cyclic points:

It was shown that the cyclic points are the intersections of the quartic with the circles of inversion. Therefore, the cyclic points on the curve are inverted into cyclic points on the inverse curve.

For, at a cyclic point four consecutive points lie on a circle. When the curve is inverted, these four consecutive points are inverted into four consecutive points on the inverse curve and the circle is inverted into a circle through four consecutive points on the inverse curve. Hence the inverse of a cyclic point is a cyclic point; and since these points lie on the circles of inversion, all the above properties, which we have established for the foci, are also true for the cyclic points.

187. Bicircular quartic with three given foci:

The locus of a point, whose distances from three fixed points are connected by the relation $l\rho+m\rho'+n\rho''=0$, *is a bicircular quartic having those three fixed points for foci.*

Take O, one of the fixed points, as origin and another point A situated on the axis of x at a distance c from the origin. Let (x', y') be the co-ordinates of the third fixed point B.

Now, if P be the point (x, y), we have—

$$\rho^2=x^2+y^2\,; \qquad \rho'^2=(x-c)^2+y^2$$

and
$$\rho''^2=(x-x')^2+(y-y')^2$$

$$\therefore\quad l\rho+m\rho'+n\rho''=l\sqrt{x^2+y^2}+m\sqrt{(x-c)\ +y^2}$$
$$+n\sqrt{(x-x')\ +(y-y')^2}=0$$

i.e.,
$$l\sqrt{u}+m\sqrt{v}+n\sqrt{w}=0 \qquad \dots \quad (1)$$

where u, v, w are point-circles, whose centres are O, A and B respectively.

Rationalising the equation, we obtain—

$$l^4u^2+m^4v^2+n^4w^2-2m^2n^2vw$$
$$-2n^2l^2wu-2l^2m^2uv=0 \qquad \ldots \quad (2)$$

which is evidently the equation of a bicircular quartic, when u, v, w are circles

Now the circle $u=0$ meets this quartic, where

$$m^4v^2+n^4w^2-2m^2n^2vw=0, \text{ i.e. } (m^2v-n^2w)^2=0$$

i.e., $u=0$ touches the quartic at the two finite points, where $m^2v-n^2w=0$ meets it. Thus the point circle $u=0$ has double contact with the bicircular quartic, or what is the same thing, the point O is a focus of the curve. Similarly, it can be shown that A and B are each a focus of the curve. We shall more fully discuss the properties of these curves in a subsequent chapter.

Ex. Show that $l\rho+m\rho'+n\rho''=0$ represents a two-circuited bicircular quartic.

188. THE FOURTH FOCUS OF THE CONCYCLIC GROUP:

We shall determine the fourth focus of the bicircular quartic which is concyclic with the three given foci.

The circle

$$\lambda u+\mu v+\nu w=0 \qquad \ldots \quad (1)$$

touches the quartic

$$l\sqrt{u}+m\sqrt{v}+n\sqrt{w}=0 \qquad \ldots \quad (2)$$

if

$$\frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{n^2}{\nu}=0 \qquad \ldots \quad (3)$$

Hence the quartic (2) may be regarded as the envelope of (1), provided the condition (3) is satisfied.

Now, (1) will be a point-circle if the discriminant vanishes, *i.e.*, if

$$(\lambda+\mu+\nu)\{\mu c^2+\nu(x'^2+y'^2)\}-\nu^2y'^2-(c\mu-\nu x')^2=0 \quad \dots \quad (4)$$

Now put $\quad OB^2 = b^2 = x'^2+y'^2$

and $\quad AB^2 = a^2 = (x'-c)^2+y'^2$

$$\therefore \quad 2cx'=a^2-b^2-c^2.$$

$\therefore$ The equation (4) becomes—

$$(\lambda+\mu+\nu)\{\mu c^2+\nu b^2\}-\nu^2b^2-\mu^2c^2-\mu\nu(b^2+c^2-a^2)=0$$

or, $$\mu\nu a^2+\lambda\nu b^2+\lambda\mu c^2=0$$

i.e., $$\frac{a^2}{\lambda}+\frac{b^2}{\mu}+\frac{c^2}{\nu}=0 \quad \dots \quad (5)$$

Therefore, when the conditions (3) and (5) are satisfied, the equation (2) represents a point-circle having double contact with the curve. Hence, solving (3) and (5) we obtain the values of λ, μ, ν, and the fourth focus is thereby determined.

189 THEOREM: *

If four concyclic foci of a bicircular quartic are given, through any point there can be described two such quartics and they cut each other at right angles.

The equation of a bicircular quartic having three given foci is—

$$l\sqrt{u}+m\sqrt{v}+n\sqrt{w}=0 \quad \dots \quad (1)$$

The fourth focus is determined by—

$$\lambda u+\mu v+\nu w=0$$

where $$\frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{m^2}{\nu}=0 \quad \dots \quad (2)$$

$$\frac{a^2}{\lambda}+\frac{b^2}{\mu}+\frac{c^2}{\nu}=0 \quad \dots \quad (3)$$

* Salmon—H. P. Curves, § 277.

When the fourth focus is given, the values of λ, μ, ν are given also. Hence, if the curve passes through any point (x', y', z'), we must have—

$$l\sqrt{u'}+m\sqrt{v'}+n\sqrt{w'}=0$$

and

$$\frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{n^2}{\nu}=0.$$

But these two equations determine two sets of values of l, m, n, when (x', y', z') is fixed. Hence, *two* curves can be described through any given point.

Let $$l\sqrt{u}+m\sqrt{v}+n\sqrt{w}=0$$

and $$l'\sqrt{u}+m'\sqrt{v}+n'\sqrt{w}=0$$

represent the two quartics. Since they are confocal, *i.e.*, have four foci common, we must have—

$$\frac{a^2}{\lambda}+\frac{b^2}{\mu}+\frac{c^2}{\nu}=0,\ \frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{n^2}{\nu}=0,\ \frac{l'^2}{\lambda}+\frac{m'^2}{\mu}+\frac{n'^2}{\nu}=0.$$

Eliminating λ, μ, ν between these three equations, we obtain—

$$a^2(m^2n'^2-m'^2n^2)+b^2(n^2l'^2-n'^2l^2)$$
$$+c^2(l^2m'^2-l'^2m^2)=0 \quad \dots \quad (4)$$

which is the condition that the two curves may be confocal.

In order to prove that the two curves cut each other orthogonally, it will be sufficient to find the condition that the generating circles of the two curves, namely,

$$\lambda u+\mu v+\nu w=0 \quad \text{and} \quad \lambda' u+\mu' v+\nu' w=0$$

cut each other orthogonally. This condition is found, after calculation, to be —

$$a^2(\mu\nu'+\mu'\nu)+b^2(\nu\lambda'+\nu'\lambda)+c^2(\lambda\mu'+\lambda'\mu)=0 \quad \dots \quad (5)$$

But the circle $\lambda u+\mu v+\nu w=0$ touches the quartic

$$l\sqrt{u}+m\sqrt{v}+n\sqrt{w}=0$$

if
$$\frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{n^2}{\nu}=0.$$

$\therefore$ If we take $\lambda : \mu : \nu = \frac{l}{\rho} : \frac{m}{\rho'} : \frac{n}{\rho''}$

the circle
$$\frac{l}{\rho}u+\frac{m}{\rho'}v+\frac{n}{\rho''}w=0 \qquad \dots \quad (6)$$

touches the quartic at the point, where $\sqrt{u}$, $\sqrt{v}$, $\sqrt{w}$ are ρ, ρ', ρ''.

Similarly,
$$\frac{l'}{\rho}u+\frac{m'}{\rho'}v+\frac{n'}{\rho''}w=0$$

touches the second quartic—

$$l'\sqrt{u}+m'\sqrt{v}+n'\sqrt{w}=0$$

at the same point. These two circles cut orthogonally, if

$$a^2 . \frac{mn'+m'n}{\rho'\rho''}+b^2\frac{nl'+n'l}{\rho\rho''}+c^2\frac{lm'+l'm}{\rho\rho'}=0 \qquad \dots \quad (7)$$

But solving the two equations—

$$l\rho+m\rho'+n\rho''=0 \quad \text{and} \quad l'\rho+m'\rho'+n'\rho''=0$$

for ρ, ρ', ρ'', we obtain—

$$\rho : \rho' : \rho''=mn'-m'n : nl'-n'l : lm'-l'm.$$

Substituting these values in (7), we find—

$$a^2(m^2n'^2-m'^2n^2)+b^2(n^2l'^2-n'^2l^2)+c^2(l^2m'^2-l'^2m^2)=0$$

which is the same condition that the two curves are confocal, whence the truth of the theorem follows.

Ex. 1. Prove that $l\rho+m\rho'+n\rho''=0$ represents a circular cubic, if $l\pm m\pm n=0$.

[The co-efficient of $(x^2+y^2)^2$ should vanish.]

Ex. 2. Show that the locus of the common vertex of two triangles whose bases are given and the vertical angles are equal is a circular cubic.

190. Bicircular quartics with a third double point:

Bicircular quartics can be divided into two classes, according as the curve has two or three nodes. The case of two nodes we have already discussed. Now we shall study some properties of a bicircular quartic having a third node in the finite part of the plane.

The inverse of a conic with respect to any point, not on the curve, is a bicircular quartic having a third double point at the centre of inversion; and this point will be an acnode, a cusp or a crunode, according as the conic is an ellipse, a parabola or a hyperbola.

The equation of a conic, referred to any point as origin, is

$$u_2+u_1+u_0=0$$

The inverse of this curve is—

$$u_0(x^2+y^2)^2+k^2u_1(x^2+y^2)+k^4u_2=0 \qquad \dots \quad (1)$$

which is evidently a bicircular quartic with a double point at the origin. It will be a node, a cusp or a conjugate point, according as u_2 represents a pair of lines, (1) real and distinct, (2) coincident, or (3) imaginary; *i.e.*, according as the given conic is a hyperbola, a parabola or an ellipse.

The inverse of the conic $x^2/a^2+y^2/b^2=1$ *w.r.t.* any point (x', y') is

$$k^4(x^2/a^2+y^2/b^2)+2k^2r^2(x'x/a^2+y'y/b^2)$$
$$+(x'^2/a^2+y'^2/b^2-1)r^4=0 \quad \dots \quad (2)$$

When the centre of inversion is a focus of the conic, the inverse is a *cartesian*. It is to be noticed, however, that in this case, the inverse of an ellipse or a hyperbola is called a *limacon* and that of a parabola is called a *cardioid*. The inverse of a rectangular hyperbola with respect to its centre is the *lemniscate* of Bernoulli, which possesses a pair of biflecnodes at the circular points.

When the origin lies on the conic, the inverse reduces to a nodal circular cubic.

191. THE PEDAL OF A CONIC :

The pedal of a central conic with respect to any point in its plane is a bicircular quartic, having a third double point at the origin, which is a node, a cusp or a conjugate point, according as the origin lies without, on or within the conic.

Let the equation of a conic be—

$$ax^2+by^2+2hxy+2fy+2gx+c=0 \qquad \dots \quad (1)$$

The condition that any line $x\cos\theta+y\sin\theta=p$ touches this conic may be written as—

$$\mathrm{A}\cos^2\theta+\mathrm{B}\sin^2\theta+\mathrm{C}p^2-2\mathrm{F}p\sin\theta-2\mathrm{G}p\cos\theta$$
$$+2\mathrm{H}\sin\theta\cos\theta=0 \qquad \dots \quad (2)$$

where A, B, C, ... are the minors of the determinant

$$\Delta\equiv\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

The co-ordinates of the foot of the perpendicular drawn from the origin are $(p\cos\theta,\ p\sin\theta)$.

Multiplying (2) by p^2 and substituting x and y for $p\cos\theta$, $p\sin\theta$, and $p^2=x^2+y^2$, we obtain—

$$\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}(x^2+y^2)^2-2(\mathrm{F}y+\mathrm{G}x)(x^2+y^2)$$
$$+2\mathrm{H}xy=0$$

i.e.,
$$\mathrm{C}(x^2+y^2)^2-2(\mathrm{F}y+\mathrm{G}x)(x^2+y^2)$$
$$+(\mathrm{A}x^2+\mathrm{B}y^2+2\mathrm{H}xy)=0$$

which is a bicircular quartic, having a third double point at the origin.

Now, the tangents at the double point are given by—

$$\mathrm{A}x^2+2\mathrm{H}xy+\mathrm{B}y^2=0$$

and these are real and distinct, coincident or imaginary,

according as $\quad AB-H^2 \quad <, \quad =, \quad \text{or} \quad > 0$

i.e., according as $\quad \Delta.c \quad <, \quad =, \quad \text{or} \quad > 0$

i.e., according as $\quad c \quad <, \quad =, \quad \text{or} \quad > 0 \qquad \ldots (3)$

Now, when the origin lies outside the conic, two real tangents can be drawn to it and the triangle formed by these tangents and their chord of contact is real. When the point lies on the conic, the area vanishes, and when it lies inside the conic, the triangle is imaginary.

The expression for area * of the triangle is—

$$\frac{c\sqrt{-\Delta.c}}{\{(ab-h^2)c-\Delta\}}$$

which is real, zero, or imaginary, according as $\Delta.c$ is negative, zero or positive, which is the same condition as (3), whence the truth of the theorem follows.

If $C=0$, *i.e.*, $ab-h^2=0$, the given conic becomes a parabola, and the curve reduces to a circular cubic. Hence, *the pedal of a parabola is a circular cubic.*

The pedal of the ellipse $x^2/a^2+y^2/b^2=1$, referred to any point (x', y') is—

$$(x^2+y^2+xx'+yy')^2=a^2x^2+b^2y^2$$

and the origin is a node, a cusp or a conjugate point according as—

$$x'^2/a^2+y'^2/b^2 \quad >, \quad =, \quad \text{or} \quad <1$$

i.e., according as the point (x', y') lies without, on, or within the ellipse.

The pedal of the parabola $y^2=4ax$ is the circular cubic—

$$(x^2+y^2+xx'+yy')x=ay^2.$$

* Sir Asutosh Mukhopadhyay—A Memoir on Plane Analytical Geometry—Journal of the Asiatic Society of Bengal, Vol. 56, Part II, No. 3 (1877), § 21.

192. BICIRCULAR QUARTICS WITH A PAIR OF FLECNODES :

We shall now consider a bicircular quartic having a pair of flecnodes at the circular points.

The equation of a quartic having flecnodes at B and C is given by equation (1) of Article 163.

If now we project B and C to the circular points at infinity, we must put—

$$x=1, \quad y=\xi+i\eta \quad \text{and} \quad z=\xi-i\eta$$

in the equation, which then becomes—

$$4h^2q(\xi^2+\eta^2)^2+p1^2(\xi-i\eta)^2/(p+q)^2+4f^2p1^2(\xi+i\eta)^2$$
$$+2 1(\xi^2+\eta^2)\{f1+2fh(p+q)(\xi+i\eta)+h(\xi-i\eta)\}=0$$

If this is to be a real curve, we must have—

$$f(p+q)=1, \quad \text{and} \quad \text{putting} \quad fq=a', \quad 1/hq=b'$$

the equation reduces to the form—

$$(\xi^2+\eta^2)^2+2b'\xi(\xi^2+\eta^2)$$
$$+a'b'^2\{(3-2a')\xi^2-(1-2a')\eta^2\}=0 \quad \dots \quad (1)$$

which, therefore, is the equation of a bicircular quartic, having a pair of flecnodes at the circular points.

The origin is evidently a node ; it will be a cusp with $\eta=0$ as the cuspidal tangent, if $2a'=3$, and the equation then reduces to—

$$(\xi^2+\eta^2)^2+2b'\xi(\xi^2+\eta^2)+3b'^2\eta^2=0 \quad \cdots \quad (2)$$

When, however, $1-2a'=0$, the equation reduces to the form—

$$(\xi^2+\eta^2+b'\xi)^2=0$$

which is the square of a circle.

If we compare the equation (1) with (2) of § 190, we find that $y'=0$, and $$2b'(a^2+b^2)=3a^2+b^2$$

whence $$(x'^2-a^2)(a^2+b^2)^2=x'^2b^2(3a^2+b^2)$$

i.e., $$x'^2(a^2-b^2)=(a^2+b^2)^2$$

Thus, the co-ordinates of the centre of inversion are—

$$x' = \pm \frac{a^2+b^2}{\sqrt{a^2+b^2}}, \qquad y'=0.$$

When, however, the conic is a parabola, the equation of the quartic becomes—

$$(\xi^2+\eta^2)^2+2b'\xi(\xi^2+\eta^2)+3b'^2\eta^2=0$$

The centre of inversion is found to be the point $(-3a, 0)$.

Ex. 1. If the circular lines through the origin are taken as the inflexional tangents of a quartic, with flecnodes at the circular points, show that the equation of the curve can be put into the form—

$$(x^2+y^2)^2+(x^2+y^2)(2gx+c)+k=0$$

where the ordinary tangents at the circular points meet at the point $(-g, 0)$.

Ex. 2. Any focal ordinate of a conic meets the director circle in a point P. If the tangent at P meets the transverse axis in Q, prove that the inverse of the conic *w.r.t.* Q as origin is a quartic with flecnodes at the circular points.

Ex. 3. Show that the inverse of a parabola, referred to a point on the opposite side of the directrix at a distance equal to that of the focus, is a bicircular quartic with a pair of flecnodes at the circular points.

Ex. 4. Prove that a unicursal quartic with two unreal flecnodes can be projected into the inverse of a conic with respect to the reflection of a focus in the corresponding directrix.

Ex. 5. A bicircular quartic has flecnodes at the circular points and a finite node. Shew that the points of contact of the tangents drawn from the node and those of the circular tangents are concyclic.

193. BICUSPIDAL QUARTICS*:

Bicuspidal quartic curves may be studied as limiting cases of binodal quartics.

The equation of a quartic with nodes at B and C can be written (§ 173) as—

$$y^2z^2+x^2(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)=0$$

the tangents at B and C being respectively—

$$z^2+bx^2=0 \quad \text{and} \quad y^2+cx^2=0$$

If these are cusps, we must have $b=0$, and $c=0$, and A is the intersection of the cuspidal tangents.

Putting $g=h=1$ and $a=f^2+2m$, the equation of the curve can then be put into the form—

$$(yz+fx^2)^2+2(mx+y+z)x^3=0 \qquad \ldots \quad (1)$$

which may again be written as—

$$\{yz+(f-\lambda)x^2\}^2$$
$$+2x^2[(f\lambda+m-\lambda^2/2)x^2+\lambda yz+zx+xy]=0 \qquad \ldots \quad (2)$$

showing that the conic—

$$(f\lambda+m-\lambda^2/2)x^2+\lambda yz+zx+xy=0 \qquad \ldots \quad (3)$$

touches the curve at the points, other than B and C, where

$$yz+(f-\lambda)x^2=0$$

meets it.

The conic (3) will degenerate into a pair of right lines, if the discriminant vanishes,

i.e., if $$\lambda(\lambda^3-2f\lambda^2-2m\lambda+2)=0 \qquad \ldots \quad (4)$$

* J. de Vries—*On bicuspidal curves of order four*—Proc. Acad., Amsterdam (1909).

The root $\lambda=0$, shows that the line $mx+y+z=0$ is a bitangent, as is evident from equation (1), the points of contact lying on the conic $yz+fx^2=0$.

In fact, the conic (3) consists of the line BC and the bitangent.

The other three roots of (4) will give the tangents w can be drawn from the cusps to the curve.

When the two cusps are at the circular points at infinity, the curve is called a Cartesian, which was originally studied by Descartes. We shall have occasion to discuss the properties of such curves in some details in a subsequent chapter of this volume.

Ex. 1. If λ_1, λ_2, λ_3 are the roots of $\lambda^3-2f\lambda^2-2m\lambda+2=0$, show that $z+\lambda_1 x=0$, $z+\lambda_2 x=0$ and $z+\lambda_3 x=0$ represent the tangents drawn from B to the curve, and $y+\lambda_1 x=0$, etc., those drawn from C.

Ex. 2. Prove that the conic through the points of contact of the bitangent and those of the tangents drawn from B touches the curve at B.

[The conic is $yz+fx^2=2z(mz+y+z)$.]

Ex. 3. Show that the points of contact of the other four tangents drawn to the curve from any point of the bitangent lie on a conic through the cusps.

Ex. 4. Prove that the eight inflexions lie on a cubic through the cusps and the intersection of the cuspidal line with the bitangent.

CHAPTER X

Circular Cubics as Degenerate Bicircular Quartics

194. The circular cubic:

We have seen that a binodal quartic reduces to a cubic and the line joining the nodes, when the focal conic touches this line. In the case of a bicircular quartic, the line joining the nodes is the line at infinity, and consequently, when the focal conic becomes a parabola, the bicircular quartic reduces to a circular cubic and the line at infinity.

A circular cubic can be regarded as the envelope of a variable circle which cuts a fixed circle orthogonally and whose centre moves along a parabola.

Let
$$y^2 = 4ax \qquad \dots \quad (1)$$
be a fixed parabola

and
$$x^2 + y^2 + 2fx + 2gy + c = 0 \qquad \dots \quad (2)$$
be a fixed circle.

Taking any point $(at^2, 2at)$ on the parabola as centre, the equation of a circle cutting the circle (2) orthogonally is*—

$$(x - at^2)^2 + (y - 2at)^2 = a^2t^4 + (2af + 4a^2)t^2 + 4agt + c$$

i.e.,
$$2a(x+f)t^2 + 4a(y+g)t = x^2 + y^2 - c \qquad \dots \quad (3)$$

Differentiating this with respect to t, we obtain—

$$4a(x+f)t + 4a(y+g) = 0$$

which gives
$$t = -(y+g)/(x+f)$$

* See § 169.

$\therefore$ The envelope of the variable circle (3) becomes—

$$2a\ \frac{(y+g)^2}{(x+f)}-4a.\frac{(y+g)^2}{(x+f)}=x^2+y^2-c$$

i.e., $$(x^2+y^2-c)(x+f)+2a(y+g)^2=0 \qquad \ldots (4)$$

which is the equation of a circular cubic and consequently, a circular cubic is a degenerate bicircular quartic. All the properties established for a bicircular quartic must then hold for a circular cubic. If the focus of the parabola be taken as the origin, the equation (4) will be transformed into the form—

$$(x+f)(x^2+y^2)+ax+by+c=0.$$

Since a bicircular quartic has four circles of inversion and four corresponding focal conics belonging to the same confocal system, the focal conics have the same real foci and axes. Consequently, when one focal conic is a parabola, the other three must be parabolas, and a circular cubic can be generated by taking any of them as focal conic. Taking the centre of the fixed circle as the origin, and $(-f,-g)$ the co-ordinates of the focus of the focal parabola, and a as the focal distance, the equation of the circular cubic is obtained in the form—

$$(x+2a)(x^2+y^2)+x(2fx+2gy+\delta^2)=0.$$

195. THE ASYMPTOTE :

The equation (4) of the cubic shows that the curve passes through the two circular points at infinity and through another point at infinity in the direction of the line $x+f=0$, which must be a real point. The line $x+f=0$ is a tangent to the curve at the centre of inversion $(-f,-g)$. But $x+f=0$ gives the direction of the real asymptote to the cubic. Therefore the tangent $x+f=0$ meets the cubic again in a real point at infinity.

Thus the four centres of inversion lie on the four tangents which can be drawn to the curve from the real point at infinity. These tangents are therefore parallel to the real asymptote.

196. SELF-INVERSE CIRCULAR CUBIC:

Taking the point of contact of one of these tangents as origin and the tangent at the origin as the axis of y, the equation of the cubic can be written in the form—

$$x(x^2+y^2)+ax^2+2hxy+by^2+kx=0$$

which remains unchanged, if we write—

$$kx/(x^2+y^2), \qquad ky/(x^2+y^2)$$

for x and y respectively, *i.e.*, if we invert the curve with respect to the circle of radius $k^{\frac{1}{2}}$ and centre at the origin.

Hence, *the cubic is self-inverse with regard to this circle.*

Since there are four tangents and four corresponding points of contact, we obtain the theorem:

A circular cubic is self-inverse with respect to four circles.

If, however, the origin is not on the cubic, the inverse is a *bi-circular* quartic. It is to be noted that the circular cubic passes through the four centres of inversion. Since the cubic is self-inverse, the system of four circles are also self-inverse with respect to any one of them. The circles are then all mutually orthogonal.

Ex. 1. Obtain the equation of a circular cubic in the form—

$$x(x^2+y^2)+ax^2+2hxy+2gx+2fy=0$$

[Take the origin at the point where the real asymptote meets the curve.]

Ex. 2. Show that the tangents to a circular cubic drawn from a point where the tangent is parallel to the real asymptote are all equal.

Ex. 3. Show the cubic— $x(x^2+y^2)+ax^2+2hxy+by^2+kx=0$ is self-inverse with respect to the circle $x^2+y^2=k$, the corresponding focal parabola being $(y+h)^2=b(2x+a)$

Ex. 4. Prove that the centres of three circles of inversion are the vertices of the common self-conjugate triangle of the focal parabola and the fourth circle of inversion.

Ex. 5. If a circular cubic be inverted from the point where the asymptote meets the curve, prove that the origin is a point of inflexion on the inverse curve.

197. REAL CIRCLES OF INVERSION:

We have shown that the four centres of inversion of a bicircular quartic are such that each is the orthocentre of the triangle determined by the other three and the triangle is self-conjugate with respect to the circle with that as centre; the radical axis of any two circles must pass through the centres of the other two. This is also true in the case of a circular cubic, but in this case, the four centres lie on the curve.

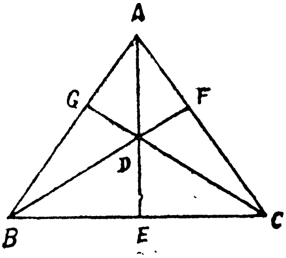
Thus, *in general, of the four circles of inversion, one is imaginary and the other three real.*

Let A, B, C, D be the four centres, and consider the triangle ABC, which is self-conjugate with respect to the circle with D as centre.

Let AD, BD, CD meet the opposite sides in E, F, G respectively. Since A is the pole of BC, we must have—

$$DA.DE = r^2, \quad \text{where } r \text{ is the radius of the circle.}$$

Now, when A and E lie on the same side of the centre D, DA and DE are of the same sign, and consequently the product DA. DE is positive, making r real. But if A and E are on opposite sides of D, DA and DE are of opposite signs, and consequently the product DA. DE is a negative quantity, and the radius r is imaginary.



Now, it is seen that if the four centres are all real, of the four triangles formed by them, one is acute-angled and the other three obtuse-angled, and in the case of an acute-angled triangle the radius becomes imaginary.

Hence, *of the four circles of inversion one is imaginary and the other three real.*

It is to be noted, however, that the curve is in this case bipartite. If the curve is unipartite, two of the circles are real and the other two have unreal centres.

If we invert the cubic with respect to a real circle of inversion whose centre is B (say), then the inverse of D will be a point on the cubic. For, since ACD is self-conjugate with respect to the circle B, $BD.BF=r^2$, and F is the inverse of D, but the cubic is inverted into itself. Hence the inverse F of D lies also on the cubic. Similarly, E and G are also points on the curve. This is not, however, true for the bicircular quartic.

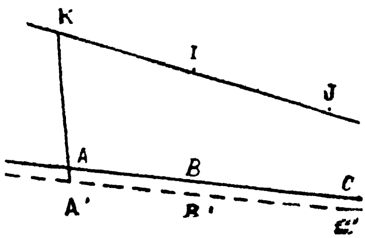
Since E, F, G are the feet of the perpendiculars of the triangle ABC, we obtain the theorem:

Every circular cubic passes through the four centres of inversion and also through the feet of the perpendiculars of the triangle formed by joining any three centres of inversion.

Ex. Prove that the tangents to a circular cubic at the centres of inversion are parallel to the asymptote.

198. THEOREM:

If a line through a centre of inversion intersects the circular cubic in two points, a circle can be drawn having simple contact with the cubic at these two points.



Let A be a centre of inversion, and let any line through A intersect the cubic in two other points B and C. Then

A is the point of contact of a tangent drawn from the real point K at infinity on the curve. Now consider a line A'B'C' consecutive to ABC.

The following three cubics then pass through the eight points A, B, C, A', B', C', K and I :—

(*i*) The given cubic ;

(*ii*) The cubic consisting of the three lines ABC, A'B'C' and KIJ ;

(*iii*) The line A'AK and a conic determined by B, C, B', C', I.

Therefore they must pass through a ninth common point J.

Hence, the six points B, C, B', C', I, J lie on a conic, which is evidently a circle ; and when ABC, A'B'C' coincide, the circle has double contact with the cubic at B and C. The same easily follows from the fact that the generating circle of a bicircular quartic has double contact with the curve, the chord of contact passing through a centre of inversion, and the circular cubic being a degenerate bicircular quartic, the truth of the proposition is manifest.

199. FOCI OF CIRCULAR CUBICS :

It has been shown that the sixteen foci of a bicircular quartic lie by fours on four circles. The same must then be true of a circular cubic.

Suppose A is one of the circles of inversion and let a tangent drawn from the circular point I to the cubic meet A at a. If now the cubic is inverted *w.r t.* the circle A, the cubic is inverted into itself and the tangent Ia is inverted into the tangent Ja, and consequently, the point a is a focus. Now, since four tangents can be drawn from I to the cubic, besides the tangent at I, and there are four corresponding tangents from J, they intersect in sixteen points, each of which is a focus of the cubic.

Hence, *the sixteen foci of a circular cubic lie by fours on four circles of inversion.*

From what has been said above, it follows that the pencil of tangents drawn from I and that drawn from J have the same cross-ratio, since the circle passes through the points I and J. Projecting the points I and J into two finite points of the plane, we obtain Salmon's theorem on the constant cross-ratio property of a cubic.*

If a is real, the other real foci will be obtained by inverting a repeatedly with respect to the remaining three circles of inversion. The circle A will remain unaltered and the four real foci lie on this.

200. THE SINGULAR FOCUS:

Suppose the four concyclic foci $\delta_1, \delta_2, \delta_3, \delta_4$ lie on the circle with centre D. Then the diagonal points A, B, C of the quadrangle $\delta_1\delta_2\delta_3\delta_4$ are the other three centres of inversion and D is the orthocentre of the triangle ABC.

Now, taking ABC for the triangle of reference, the equation of a conic through $\delta_1, \delta_2, \delta_3, \delta_4$ may be written as

$$lx^2+my^2+nz^2=0 \qquad \dots \quad (1)$$

If this is to be a parabola, we must have—

$$a^2/l+b^2/m+c^2/n=0 \qquad \dots \quad (2)$$

where a, b, c are the lengths of the sides of the triangle ABC.

Now, if E, F, G be the mid-points of the sides of the triangle ABC, the equation of the line EF is $by+cz-ax=0$, which evidently touches the parabola, in virtue of the relation (2).

Similarly, FG and EG touch the same parabola.

Hence, the focal parabola is inscribed in the triangle EFG and its focus lies on the circle circumscribing the

* See § 51.

triangle EFG, *i.e.*, the nine-points circle of the triangle ABC.

Hence, the focus of the focal parabola lies on the nine-points circle of the triangle formed by the three centres of inversion. But the four triangles formed by the four centres A, B, C, D of inversion, taken three at a time, have a common nine-points circle. Hence we obtain the theorem :—

The singular focus of the circular cubic lies on the common nine-points circle of the triangles formed by any three of the four centres of inversion.

Again, since the focal parabola touches the sides of the triangle formed by joining the mid-points E, F, G of the triangle ABC, the directrix of the parabola passes through the orthocentre of the triangle EFG.* But the orthocentre of EFG is the circumcentre of the triangle ABC. Hence the directrix passes through the circumcentre of the triangle ABC. Thus we obtain the theorem :—

The directrices of the focal parabolas respectively pass through the centres of the circles circumscribing the triangles formed by the centres of inversion, taken three at a time.

Ex. 1. Find the singular focus of the cubic—

$$x(x^2+y^2)+ax^2+2hxy+2gx+2fy=0$$

[The singular focus is the point— $(-a/2,-h)$.]

Ex. 2. The normals to a circular cubic at its intersections with a circle of inversion touch a parabola with focus at the singular focus and axis perpendicular to the asymptote, *i.e.*, the corresponding focal parabola.

Ex. 3. The cross-ratio of the pencil subtended by four concyclic foci at any point of the focal parabola through them is equal to that of the four corresponding directrices.

Ex. 4. Prove that circular cubics with the same real asymptote, singular focus, and self-inverse with respect to the same point have a common focal parabola.

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* Salmon—Conics, § 297, *Ex.* 14.

201. Circular cubics with given foci:

Suppose the focal parabola intersects the circle of inversion in the four foci a_1, a_2, a_3, a_4. Then the chords a_1a_3 and a_2a_4 are equally inclined to the axis of the parabola, *i.e.*, the axis bisects the angle between the two chords.* Since there are two bisectors mutually perpendicular, in the pencil of conics through the four points, there are *two* parabolas, whose axes are mutually orthogonal.

Hence, *given four concyclic foci, there are two circular cubics, the axes of whose focal parabolas are mutually orthogonal.*

This is a particular case of the theorem in article 189, which was originally obtained from geometrical considerations by Dr. Hart. The two cubics have seven common points, which can be geometrically determined with the help of article 13.

Ex. 1. Four concyclic foci are given. Show that there are two circular cubics whose real asymptotes are perpendicular.

Ex. 2. The line-pairs joining four concyclic foci of a circular cubic are equally inclined to the real asymptote.

Ex. 3. A conic passes through four given concyclic points. Prove that the locus of the foci is a pair of circular cubics with the given points for foci.

202. Intersections with a circle:

If a circle intersects a circular cubic in four points, the opposite sides of the quadrangle formed by these four points intersect the cubic again in three pairs of points. The line joining each pair of these points is parallel to the asymptote.

Let the circle intersect the cubic in four points A, B, C, D. Let the pairs of opposite sides AB, CD; AC, BD; BC, AD intersect the cubic again in the three pairs of points E, F; E′, F′; E″, F″ respectively. Then the lines EF, E′F′, E″F″ are parallel to the asymptote of the cubic.

* Salmon—Conics, § 244.

Taking ABC as the triangle of reference, the equation of a circle and a conic through the points A, B, C, D may be taken respectively as—

$$\mathrm{T} \equiv ayz + bzx + cxy = 0$$

and $$\mathrm{S} \equiv fyz + gzx + hxy = 0$$

The cubic is then represented by $u_1 \mathrm{T} = \mathrm{IS}$ ... (1)

where $$u_1 \equiv lx + my + nz$$

The equation of CD is obtained, by eliminating z between T and S, in the form—

$$c(fy + gx) = h(ay + bx) \qquad \ldots \quad (2)$$

Substituting in (1) from (2), we obtain—

$$(cu_1 - h\mathrm{I})(ayz + bzx + cxy) = 0$$

showing that the line $cu_1 - h\mathrm{I} = 0$ ... (3)

passes through the point F, where CD meets the cubic again.

Since, by putting $z = 0$ in (1) and (3) we obtain the same equation, it is clear that the line (3) also passes through the point E where AB meets the cubic, *i.e.*, (3) is the equation of EF.

But $cu_1 - h\mathrm{I}$ is evidently parallel to the asymptote of the cubic, showing that EF is parallel to the asymptote. Similarly, E′F′, E″F″ are found to be parallel to the same asymptote.

It follows that if the sides GE, EF and GF of the triangle EFG in § 199 be produced to meet the cubic in F′, G′, E′ respectively the lines GG′, FF′, EE′, are parallel to the asymptote.

For, the four points A,G,E,C are the intersections of the cubic and a circle, and AC, EG meet the cubic again in F and F′, FF′ is parallel to the asymptote.

The following particular cases should be noticed :

I. If A and B coincide, the circle touches the cubic at A, the line AE being the tangent, and we obtain the theorem :—

If a circle touch a circular cubic at A and intersect it in two points C and D, the line joining the tangential of A to the third point where CD cuts the cubic again is parallel to the asymptote.

II. If A, B as well as C, D coincide, the circle has double contact with the cubic, and we obtain :—

If a circle has double contact with a circular cubic, the line joining the tangentials of the points of contact is parallel to the asymptote and the chord of contact intersects the curve at a third point where again the tangent is parallel to the asymptote, i.e., the point is a centre of inversion.

III. If the three points A, B, C coincide, the circle becomes the circle of curvature at A, and the line AD is the chord of curvature. Hence we obtain the theorem :

The line joining the tangential of a point A and the third point where the chord of curvature at A intersects the cubic again is parallel to the asymptote.

IV. If the four points A, B, C, D coincide, the point A becomes a cyclic point, and the theorem becomes :—

The tangent at a cyclic point of a circular cubic intersects the cubic again in a centre of inversion.

Since the tangent at the tangential of the cyclic point is parallel to the asymptote, that point must be a centre of inversion. Thus, the cyclic points are the points of contact of the tangents drawn from the centres of inversion, and consequently there are *sixteen* cyclic points.

203. The inverse of a circular cubic:

We have shown that the inverse of a bicircular quartic is a circular cubic, when the origin is a point on the curve. Conversely, when a circular cubic is inverted from any point not on the curve, the inverse is a bicircular quartic, but if the origin is a point on the curve, it inverts into another circular cubic.

Let
$$u_1 r^2 + v_2 + v_1 + v_0 = 0$$

be the equation of a circular cubic. The inverse of this with respect to the origin is—

$$\frac{k^2}{r^2} u_1 \cdot \frac{k^4}{r^4} \cdot r^2 + \frac{k^4}{r^4} v_2 + \frac{k^2}{r^2} v_1 + v_0 = 0$$

or,
$$v_0 r^4 + k^2 r^2 v_1 + k^4 (v_2 + k^2 u_1) = 0$$

which passes through the origin and is a bicircular quartic.

If the cubic passes through the origin, $v_0 = 0$ and the inverse curve becomes—

$$r^2 v_1 + k^2 v_2 + k^4 u_1 = 0,$$

which is another circular cubic through the origin; and the two curves are such that the tangent to one at the origin is parallel to the asymptote of the inverse.

Again, if the line $u_1 = 0$ is the asymptote, then u_1 must be a factor of v_2, so that $v_2 = u_1 u'_1$. Then the inverse curve is—

$$r^2 v_1 + k^2 u_1 u'_1 + k^4 u_1 = 0$$

which shows that the origin is a point of inflexion on the inverse cubic. Hence we obtain the theorem :—

If a circular cubic be inverted from the point where the asymptote cuts the curve, the point will be a point of inflexion on the inverse curve and the asymptote will invert into the inflexional tangent.

This also follows from the following geometrical consideration :—

Since the asymptote passes through the origin, its inverse will be a line through the origin, and the origin inverts into itself. The two consecutive points of contact at infinity of the asymptote invert into two consecutive points at the origin on the inverse of the asymptote. Hence, three consecutive points at the origin lie on a right line, which is therefore a point of inflexion.

Ex. 1. Show that the osculating circle at the origin of inversion on a circular cubic inverts into the asymptote of the inverse.

Ex. 2. Prove that through any point on a circular cubic (or bi-circular quartic) can be described nine circles elsewhere osculating the curve, and of these circles three will be real and their points of contact will lie on a circle passing through the given point.

[Invert a circular cubic and its nine inflexions.]

Ex. 3. Prove that the centroid of four concyclic foci of a circular cubic is the foot of the perpendicular drawn from the singular focus on the real asymptote.

Ex. 4. Show that the curve $x(x^2+y^2)+ax^2+by^2+cx=0$ is self-inverse with respect to the circle $x^2+y^2=c$.

204. THEOREM :

The points of contact of the tangents, which can be drawn from the point where the asymptote meets the curve, lie on a circle.

From a point of inflexion, there can be drawn three tangents to a circular cubic, the points of contact lying on the harmonic polar. If now we invert with respect to the point of inflexion, the inverse will be a circular cubic, the inflexional tangent inverting into the asymptote and the tangents will invert into the tangents, while the harmonic polar will invert into a circle through the origin, which proves the proposition.

205. THEOREM :

The common nine-points circle of the triangles formed by joining any three of the centres of inversion passes through the point where the cubic is cut by the asymptote.

Suppose the nine-points circle meets the cubic in a fourth point O (Fig. §. 197) and let FO cut the cubic again in F''. Then F'F'' and FF' are both parallel to the asymptote by § 202. Hence F and F'' coincide and FO is the tangent to the cubic at F. Similarly, EO and GO are tangents at E and G respectively. We may then state the following theorem :

The tangents at E, F, G meet at the point O where the nine-points circle cuts the cubic again.

Since the points E, F, G lie on the nine-points circle which also passes through O, the point O is the intersection of the cubic with its asymptote, which proves the proposition.

It is then evident that the four centres of inversion A, B, C, D are the centres of the three escribed and the inscribed circles of the triangle EFG, where E, F, G are the points of contact of the tangents drawn from the point O where the asymptote cuts the cubic.

Ex. 1. Show that the equation of the cubic can be put into the form—

$$S(lx\cos A + my\cos B + nz\cos C)$$
$$= I(lx^2\cos^2 A + m\cos^2 By^2 + n\cos^2 Cz^2)$$

where $S \equiv a\cos Ax^2 + b\cos By^2 + c\cos Cz^2$ and $l+m+n=0$.

[Take ABC as the triangle of reference.]

Ex. 2. Show that the tangent at A is—

$$y(m\sin C + n\cos A\sin B) + z(n\sin B + m\cos A\sin C) = 0$$

which is parallel to the asymptote.

206. POINTS OF INFLEXION ON A CIRCULAR CUBIC :

Let the equation of the curve be taken in the form—

$$(lx+my+nz)(ayz+bzx+cxy)$$
$$=(ax+by+cz)(a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2'hxy)$$

If the curve passes through B and C, we must have—

$b'=c'=0$, and the equation reduces to—

$$(lx+my+nz)(ayz+bzx+cxy)$$
$$=(ax+by+cz)(ax^2+2f'yz+2g'zx+2h'xy) \quad \ldots \quad (1)$$

Suppose the vertices B and C are points of inflexion on the curve, with BA and CA as the inflexional tangents.

Therefore, if we put $y=0$ or $z=0$ in the equation (1) it should reduce to $x^3=0$.

Putting $y=0$, we obtain—

$$b(lx+nz)zx=(ax+cz)(a'x^3+2g'zx)$$

whence, we must have—

$$bl-ca'-2ag'=0 \quad \text{and} \quad bn-2cg'=0$$

$$\therefore \quad ca'=bl-2ag'=b(cl-an)/c \quad \ldots \quad (2)$$

Similarly, putting $z=0$, we obtain—

$$ba'=c(bl-am)/b \quad \ldots \quad (3)$$

Since the third real inflexion D must lie on the line BC, we obtain the equation of the line AD, by putting $x=0$ in (1), in the form—

$$(my+nz)a=2f'(by+cz)$$

If, however, D be at infinity, AD is parallel to BC, and we must have $m/b=n/c$

Hence from (2) and (3) we obtain—

$$a'c^2 = bcl - abn = bcl - acm = a'b^2$$

whence $\qquad a'(c^2 - b^2) = 0, \quad i.e., \qquad c = b$

and consequently $\qquad m = n.$

The asymptote is thus given by $lx + m(y+z) = 0$, which is a line parallel to BC.

We thus obtain the theorem :

If a circular cubic has three real inflexions, one of which is at infinity, the tangents at the finite inflexions form an isosceles triangle with the inflexional line as base, and the inflexional line is parallel to the asymptote of the curve.

Ex. 1. P and Q are two points on a circular cubic such that the osculating circle at P passes through Q and *vice versâ*, prove that the tangents at those points meet on the cubic.

Ex. 2. The radius of curvature of a circular cubic is a maximum or a minimum at its intersections with the circles of inversion.

Ex. 3. A chord is drawn through the point where the real asymptote meets a circular cubic. Prove that the intersections are equidistant from the singular focus.

CHAPTER XI

SPECIAL CURVES OF ORDER FOUR

207. SPECIAL QUARTICS:

We shall, in the present chapter, discuss the properties of a variety of well-known curves of the fourth order, which have acquired historical importance, having been associated, for the most part, with the names of some of the greatest geometers of past ages. All the projective properties of these curves can be easily generalised by projection, as we have already shown in the case of tricuspidal or tribiflecnodal curves.

208. THE CASSINIAN:

The locus of a point which moves so that the product of its distances from two fixed points is constant is called a Cassini's oval or the *Cassinian*. This curve was originally studied by Dominicus Cossini in connection with problems in astronomy and is accordingly named after him.*

In order to obtain the Cartesian equation of the curve, we take the middle point O of the line joining the two fixed points F_1 and F_2 as the origin of a system of rectangular axes, and OF_2 as the axis of x. If we denote the distance between F_1 and F_2 by $2a$ and the constant by c^2, the locus of a point $P(x, y)$ on the curve is given by—

$$\sqrt{(x-a)^2+y^2}.\sqrt{(x+a)^2+y^2}=c^2$$

or $$(x^2+y^2)^2-2a^2(x^2-y^2)+a^4-c^4=0 \qquad \ldots \quad (1)$$

* Cassini—*Eléments d'astronomie*—(1749), p. 149.

which may again be written in the form—

$$(x^2+y^2+a^2)^2-4a^2x^2-c^4=0 \qquad \ldots \quad (2)$$

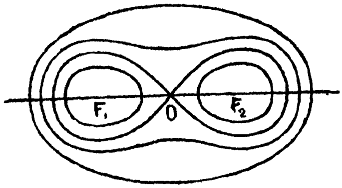
or, putting $c^4=4a^2k^2$, in the form—

$$(x^2+y^2+a^2)^2=4a^2(x^2+k^2)$$

showing that the curve is a bicircular quartic, and is symmetrical with respect to the two mutually perpendicular axes.

The equation of the curve in polar co-ordinates can be put into the form*:

$$r^4-2a^2r^2\cos 2\theta+a^4-c^4=0 \qquad \ldots \quad (3)$$



The curve always cuts the axis of x in two real points—

$$x=\pm\sqrt{a^2+c^2}$$

and in two other points—

$$x=\pm\sqrt{a^2-c^2}$$

which are real, only when $a^2>c^2$. The axis of y meets it in two imaginary points and in two other points which are real, only when $a^2<c^2$. Thus it is seen that if $a^2>c^2$, the Cassinian consists of two detached ovals, each of which encloses one of the fixed points F_1 and F_2. If $a^2<c^2$, it consists of a single circuit. The forms of the Cassinian are shown in the diagram.

When $a^2=c^2$, the curve becomes a lemniscate of Bernoulli.

* For detailed information the student is referred to Garlin—*Nouvelles An. de Mat,* (1855), p. 305; Darboux—Sur une classe remarkáble de courbes, etc., Paris (1873), p. 78.

Writing the equation (1) in the form—

$$\{x^2-(a^2-y^2)\}^2-(c^4-4a^2y^2)=0$$

it is seen that if we put $c^4=4a^2y^2$, the equation reduces to a perfect square. Consequently, the lines $\pm y=c^2/2a$ are two bitangents of the curve, the abscissæ of the points of contact being—

$$x=\pm\frac{\sqrt{4a^4-c^4}}{2a}$$

which will be real, only if $c < a\sqrt{2}$.

When $c=a\sqrt{2}$, the two points of contact coincide, and we obtain a point of undulation.

In order to determine the nature of the points at infinity, we take the circular lines and the line at infinity as the sides of the triangle of reference, *i.e.*, we put—

$$x+iy=\alpha/\gamma \qquad \text{and} \qquad x-iy=\beta/\gamma$$

and the equation (1) takes the form—

$$(\alpha^2-a^2\gamma^2)(\beta^2-a^2\gamma^2)-c^4\gamma^4=0 \qquad \dots \quad (4)$$

which shows that the lines

$$\alpha\pm a\gamma=0, \quad i.e., \quad x+iy\pm a=0$$

are tangents at the circular point ($\alpha=0$, $\gamma=0$), each meeting the curve in four coincident points, *i.e*, they are inflexional tangents.

Similarly, the lines $\beta\pm a\gamma=0$, or $x-iy\pm a=0$ are inflexional tangents at the other circular point ($\beta=0$, $\gamma=0$).

Hence, *the circular points are biflecnodes on the Cassinian.**

* Cayley—*Note on the Cassinian*—Mess. of Math., Vol. 4 (1875).

Ex. 1. ABCD is a parallelogram. The locus of a point P such that PA.PC : PB.PD is constant is a Cassinian curve.

Ex. 2. Prove that the locus of the intersections of two circles touching a tangent and having their centres at the foci of a given conic is a Cassinian curve.

Ex. 3. Show that the locus of the foci of an ellipse through a fixed point and having a given director circle is a Cassinian curve.

Ex. 4. Show that the pedal equation of the Cassinian is—

$$r^4-2a^2pr=a^4-c^4.$$

Ex. 5. Given base of a triangle $2c$ and rectangle under sides c^2. Show that the locus of vertex is a Cassini's oval, whose equation, referred to the middle point of the base as origin is—

$$(x^2+y^2-c^2)^2-4c^2x^2=a^2.$$

Ex. 6. Prove that the Cassinian curve is the envelope of a variable circle whose centre moves along a hyperbola and which cuts the director circle of the hyperbola orthogonally.

209. Other forms of equation of the Cassinian:

If the two fixed points F_1 and F_2 are taken as the poles, the curve may be represented in bipolar co-ordinates by the equation—

$$\rho\rho'=\text{a constant}=k^2 \qquad \dots \quad (1)$$

The equation of the curve may further be expressed in terms of the angle θ between the two radii ρ, ρ' joining any point P on the curve to the two fixed points F_1 and F_2 and the radius vector drawn from the mid-point O to the point P.

For, we have—

$$F_1F_2^2=PF_1^2+PF_2^2-2PF_1.PF_2\cos\theta$$

$$PF_1.PF_2=k^2, \quad PF_1^2+PF_2^2=2a^2+2OP^2.$$

$$\therefore \quad 4a^2=2a^2+2\rho^2-2k^2\cos\theta$$

i.e.

$$\rho^2=a^2+k^2\cos\theta \qquad \dots \quad (2)$$

210. FOCI OF THE CASSINIAN:

The biflecnodal tangents meet at the two fixed points $(\pm a, 0)$, which are consequently the singular foci of the curve; in fact, they are the two triple foci.

In order to determine the ordinary foci, we have to find the tangents which can be drawn from the circular points.

Consider the line $\alpha = k\gamma$ which meets the curve (4) of article 208 in four points obtained by putting $\alpha = k\gamma$ in (4), *i.e.*, the points are given by the equation—

$$\gamma^2\{(k^2-a^2)(\beta^2-a^2\gamma^2)-c^4\gamma^2\}=0$$

The line will meet the curve in two coincident points, if the expression within the bracket is a perfect square, which requires that—

$$ak=\pm\sqrt{a^4-c^4}$$

so that the equation of the tangent becomes—

$$a\alpha=\pm\sqrt{a^4-c^4}, \quad i.e., \quad x+iy=\pm\sqrt{a^4-c^4}\ /\ a$$

Similarly, the tangent drawn from the other circular point is—

$$x-iy=\pm\sqrt{a^4-c^4}\ /\ a$$

These two conjugate imaginary tangents intersect at the two real points—

$$x=\pm\sqrt{a^4-c^4}\ /\ a, \qquad y=0, \qquad \text{if} \quad a^2>c^2$$

and at the two imaginary points—

$$x=0, \qquad y=\pm\sqrt{a^4-c^4}\ /\ a, \qquad \text{if} \quad a^2<c^2$$

Hence, the ordinary foci lie on the line F_1F_2 joining the two triple foci, when the curve consists of two ovals, while they lie on the line through the mid-point O at right angles to F_1F_2, when the curve has a single circuit.

Ex. 1. Show that the orthogonal trajectories of a family of Cassinian curves with singular foci F_1 and F_2 are rectangular hyperbolas with F_1F_2 as diameter.

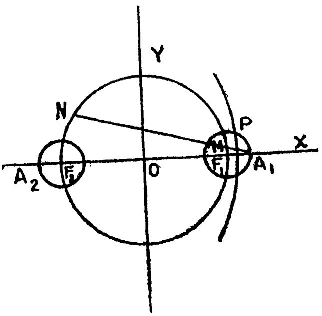
Ex. 2. The polar conic of a point P with respect to a given Cassinian curve has a constant eccentricity. Prove that the locus of P is a Cassinian curve with the singular foci of the given curve as ordinary foci.

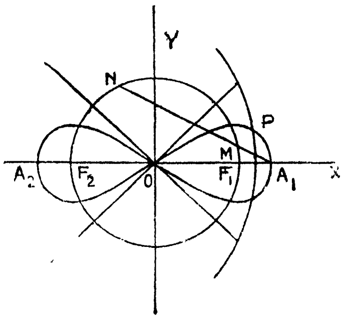
Ex. 3. Find the locus of inflexions of a system of Cassinian curves with given ordinary foci.

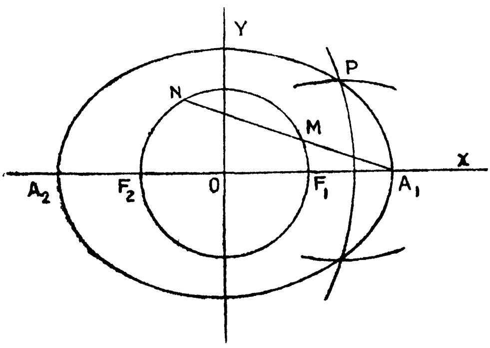
211. Construction of the Cassinian :

The Cassinian curve can be geometrically constructed in the following manner :

With centre O and radius r describe a circle. Let A_1, A_2 be two points on its diameter F_1F_2 produced, equidistant from O. Through A_1 (or A_2) draw any right line intersecting the circle in two points M and N. If now we describe two circles with F_1 and F_2 as centres and A_1M, and A_1N







respectively (or A_2M, A_2N) as radii, intersecting each other in points P, the locus of P will be a Cassinian curve.

Let O be the origin and F_1F_2 the axis of x. If d be the distance of A_1 and A_2 from O, and ω the angle which the variable line makes with the axis of x, then

$$A_1M = d\cos\omega + \sqrt{r^2 - d^2\sin^2\omega}$$

and $$A_1N = d\cos\omega - \sqrt{r^2 - d^2\sin^2\omega}$$

Then the equations of the two circles are respectively--

$$x^2 + y^2 - 2rx = d^2(\cos^2\omega - \sin^2\omega) - 2d\cos\omega\sqrt{r^2 - d^2\sin^2\omega}$$

$$x^2 + y^2 + 2rx = d^2(\cos^2\omega - \sin^2\omega) + 2d\cos\omega\sqrt{r^2 - d^2\sin\omega}$$

Eliminating ω between these two equations, we shall obtain the locus of P.

The two equations may be replaced by the following equivalent forms:

$$x^2 + y^2 = d^2(\cos^2\omega - \sin^2\omega)$$

and $$rx = d\cos\omega\sqrt{r^2 - d^2\sin^2\omega}$$

From the first of these equations we obtain—

$$\cos^2\omega = \frac{x^2 + y^2 + d^2}{2d^2}, \quad \sin^2\omega = \frac{d^2 - x^2 - y^2}{2d^2}$$

Substituting these in the second, we obtain the required locus given by—

$$(x^2 + y^2)^2 - 2r^2(x^2 - y^2) = d^2(2r^2 - d^2) \quad \ldots \quad (1)$$

which is evidently a Cassinian curve.

Since the equation remains unchanged when the sign of d is changed, it follows that the same curve will be obtained, if instead of the point A_1 we take the point A_2.

The equation (1) will be identical with (1) of § 208,

if we put $$r = a, \quad \text{and} \quad d = \sqrt{a^2 + c^2}.$$

Hence, the line joining the singular foci is a diameter of the fixed circle and the two fixed points in which the Cassinian cuts this diameter are always real. The curve is unipartite or bipartite, according as $d >, =,$ or, $< r\sqrt{2}$.

212. THE TANGENT AND THE NORMAL:

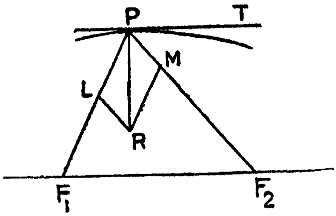
Several methods have been given for constructing the tangents and normals of the Cassinian. The following geometrical method for constructing the normal is important.*

Let the curve be defined in vectorial co-ordinates by the equation—

$$f \equiv \rho\rho' = k^2 \qquad \dots \quad (1)$$

Then, $$\frac{\partial f}{\partial \rho} \cdot \frac{\partial \rho}{\partial s} + \frac{\partial f}{\partial \rho'} \cdot \frac{\partial \rho'}{\partial s} = 0$$

Now, Let PT be the tangent at P, and $\psi_1 = \angle TPF_1$ and $\psi_2 = \angle TPF_2$.



Then, $$\frac{\partial \rho}{\partial s} = \cos\psi_1, \qquad \frac{\partial \rho'}{\partial s} = \cos\psi_2,$$

$$\therefore \qquad \frac{\partial f}{\partial \rho} \cdot \cos\psi_1 + \frac{\partial f}{\partial \rho'} \cos\psi_2 = 0 \qquad \dots \quad (2)$$

Again, from any point R on the normal, draw RL and RM respectively parallel to F_2P and F_1P.

Then, $$PL : LR = \sin RPM : \sin RPL$$
$$= \cos\psi_2 : -\cos\psi_1$$
$$= \frac{\partial f}{\partial \rho} : \frac{\partial f}{\partial \rho'}.$$

* This method may conveniently be applied for finding the normal to any curve defined in vectorial co-ordinates by an equation of the form $f(\rho, \rho') = \text{constant}$.

Hence, if we measure on PF_1 and PF_2 lengths PL and PM which are in the proportion of

$$\frac{\partial f}{\partial \rho} : \frac{\partial f}{\partial \rho'}, \quad i.e., \quad PF_2 : PF_1$$

then the diagonal of the parallelogram thus formed is the required normal at P.

Again, let $\phi_1 = \angle RPF_1$ and $\phi_2 = \angle RPF_2$

Then
$$\frac{\rho}{\sin\phi_1} = \frac{\rho'}{\sin\phi_2}.$$

At F_1, F_2 draw two lines F_1T_1 and F_2T_2 at right angles to PF_1, PF_2 respectively, meeting PT, drawn perpendicular to PR, in T_1 and T_2 respectively.

Hence,
$$PT_1 = \frac{\rho}{\sin\phi_1}, \qquad PT_2 = \frac{\rho'}{\sin\phi_2}$$

and consequently
$$PT_1 = PT_2.$$

Thus we obtain the following construction for the tangent given by Steiner*:

At F_1 and F_2 draw two lines at right angles to the respective radii. The line through P, the intercept on which by those lines is bisected at P, is the tangent at P.

Ex. 1. If PN be the normal at P on a Cassinian, prove that—

$$\angle F_1PO = \angle F_2PN.$$

Ex. 2. If the normal at P meet F_1F_2 in G, prove that

$$F_1P^2 : F_2P^2 :: F_1G : F_2G.$$

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* Steiner—*Einfache Konstruction der Tangente an die allgemeine Lemniscate*—Crelle, Bd. 14 (1835).

213. POINTS OF INFLEXION *:

The polar equation of the Cassinian is—

$$r^4-2a^2r^2\cos 2\theta+a^4-c^4=0 \qquad \ldots \quad (1)$$

If the angle which the tangent makes with the radius vector to the point of contact be denoted by ψ, we have—

$$\sin\psi=r\frac{\partial\theta}{\partial s} \quad \text{and} \quad \cos\psi=\frac{\partial r}{\partial s}.$$

Differentiating the equation (1) with respect to s, we obtain—

$$r^2\cos\psi=a^2\cos(2\theta-\psi) \qquad \ldots \quad (2)$$

Eliminating r between the equations (1) and (2), we obtain—

$$a^2\sin 2\theta=c^2\cos\psi \qquad \ldots \quad (3)$$

Differentiating this with respect to s, we obtain—

$$2a^2\cos 2\theta\frac{\partial\theta}{\partial s}=-c^2\sin\psi\frac{\partial\psi}{\partial s}$$

$$\therefore \quad \frac{\partial\psi}{\partial s}=-\frac{2a^2}{c^2}\frac{\cos 2\theta}{\sin\psi}\cdot\frac{\partial\theta}{\partial s}=-\frac{2a^2}{c^2}\cdot\frac{\cos 2\theta}{r}.$$

If ϕ is the angle which the tangent makes with the initial line, we have $\theta=\phi+\psi$.

If ρ is the radius of curvature, we have—

$$\frac{1}{\rho}=\frac{\partial\phi}{\partial s}=\frac{\partial\theta}{\partial s}-\frac{\partial\psi}{\partial s}=\frac{\partial\theta}{\partial s}+\frac{2a^2}{c^2}\frac{\cos 2\theta}{r}$$

$$=\frac{\sin\psi}{r}+\frac{2a^2\cos 2\theta}{c^2 r} \qquad \ldots \quad (4)$$

From (2) and (3) we obtain—

$$r^2\cos\psi=a^2(\cos 2\theta\cos\psi+\sin 2\theta\sin\psi)$$

$$=a^2\left(\cos 2\theta\cos\psi+\frac{c^2\cos\psi\sin\psi}{a^2}\right)$$

$$\therefore \quad r^2=a^2\cos 2\theta+c^2\sin\psi$$

* F. G. Teixeira—Tratado de las Curvas Especiales Notables (1905), p. 111.

Combining this with the equation (1) of the curve, we obtain—

$$\cos 2\theta = \frac{r^4+a^4-c^4}{2a^2r^2} \qquad \text{and} \qquad \sin\psi = \frac{r^4-a^4+c^4}{2c^2r^2}$$

$$\therefore \quad \frac{1}{\rho} = \frac{r^4-a^4+c^4}{2c^2r^3} + \frac{2a^2}{c^2r} \cdot \frac{r^4+a^4-c^4}{2a^2r^2}$$

$$= \frac{3r^4+a^4-c^4}{2c^2r^3} = \frac{3r^4-r^4+2a^2r^2\cos 2\theta}{2c^2r^3}$$

$$= \frac{2r^4+2a^2r^2\cos 2\theta}{2c^2r^3} = \frac{r^2+a^2\cos 2\theta}{c^2r}.$$

At a point of inflexion, $\rho = \infty$, and consequently

$$r^2 + a^2 \cos 2\theta = 0$$

which is a lemniscate of Bernoulli, *i.e.*, the inflexions of the Cassinian are its intersections with this curve, and since this latter equation is independent of the parameter c, it follows that all Cassinian curves with the same singular foci have their inflexions on this confocal lemniscate.

The first polar of the origin breaks up into the line at infinity and the hyperbola -

$$a^2(x^2-y^2) = a^4 - c^4$$

Eliminating c between this and the equatior (1) of the curve, we find—

$$r^2 - a^2 \cos 2\theta = 0$$

Hence we obtain the following theorem :

*The points of contact of tangents to all Cassinians with the same singular foci drawn from the mid-point of the focal line lie on a lemniscate of Bernoulli.**

* The rectification of Cassinian curves involves elliptic integrals, for which the student is referred to F. G. Teixeira—*Tratado de las curvas*, etc. (1905), pp. 113-118.

214. The Cartesian Ovals:

We have seen that a bicuspidal quartic having two cusps at the circular points at infinity is called a *Cartesian*. Descartes studied this curve and it is known after him as the *Oval of Descartes*. He defined it as the locus of a point P whose distances from two fixed points F_1 and F_2 are connected by the relation—

$$l\rho \pm m\rho' = n$$

Descartes gave the following construction of the curve:

Let F_1 and F_2 be two given points and r any line meeting F_1F_2 in A. With centre F_1 and any radius describe a circle, and let B be one of its intersections with F_1F_2. On r we now take a point C such that

$$AC : AB = \text{a constant } \lambda$$

On r we also measure $AR = AF_2$, and describe a second circle with F_2 as centre and CR as radius, which meets the first circle in P. Then the locus of P is a Cartesian oval.

We have—

$$F_1P = F_1B = F_1A + AB = F_1A + AF_2/\lambda$$

$$F_2P = RC = AC - AR = AC - AF_2$$

$$\therefore \quad \lambda.F_1P - F_2P = \lambda.AF_1 - AF_2$$

or, putting $\lambda = -\mu/\nu$, $\mu F_1P + \nu F_2P = \mu AF_1 - \nu AF_2$

$$\therefore \quad \mu.AF_1 - \nu.AF_2 = l = \mu PF_1 + \nu PF_2$$

which is certainly of the form—

$$l\rho \pm m\rho' = n \qquad \ldots \quad (1)$$

It is easily seen that the locus passes through the point A.

The bi-polar equation (1), when expressed in Cartesian co-ordinates, reduces to one of the fourth degree, and consequently represents a quartic curve.

The following particular cases should be noticed :

(1) When $l=m$, the equation becomes $\rho \pm \rho' =$constant, which evidently represents an ellipse or a hyperbola.

(2) If $n=0$, we obtain $\rho : \rho' =$constant, which represents a circle. In fact, when any one of the three quantities l, m, n vanishes, the equation represents a circle.

(3) When $l=n$ or $m=n$, the locus is a limaçon. It is easily seen that the curve is bipartite, consisting of two ovals one lying inside the other, the former corresponds to the equation $l\rho + m\rho' = n$ and the latter to $l\rho - m\rho' = n$.

215. Newton's method :

Newton defined a Cartesian oval as the locus of a point whose radial distances from two fixed circles are in a constant ratio.

If (x_1, y_1) and (x_2, y_2) be the co-ordinates of the points F_1 and F_2, then the equation (1) can be written as—

$$l\sqrt{(x-x_1)^2+(y-y_1)^2} \pm m\sqrt{(x-x_2)^2+(y-y_2)^2} = n \quad \ldots(2)$$

Now, we can determine in an infinite number of ways two constants r_1 and r_2 such that—

$$lr_1 + mr_2 = n$$

The equation (2) can then be written in the form—

$$\frac{\sqrt{(x-x_1)^2+(y-y_1)^2}-r_1}{\sqrt{(x-x_2)^2+(y-y_2)^2}-r_2} = \pm\frac{m}{l} \qquad \ldots \quad (3)$$

The numerator of this represents the distance of a point (x, y) from a circle with centre $F_1(x_1, y_1)$ and radius r_1, while the denominator represents the distance of the same point from the circle with centre $F_2(x_2, y_2)$ and radius r_2, whence the truth of the statement follows.

Chasles has given a second method of generating the curve similar to that of Newton.

Let C_1 and C_2 be two circles with centres O_1 and O_2 and O the middle point of O_1O_2. Suppose a radius vector drawn through O meets the circles in P_1, P_2 and Q_1, Q_2 respectively. Then the radii O_1P_1, O_1P_2 of C_1 meet the radii O_2Q_1, O_2Q_2 of C_2 in four points P, whose locus is a Cartesian oval. Consider the triangle O_1O_2P which is met by the variable line in O, P_1, Q_1. Then, by the theorem of Menelaus, we must have—

$$\frac{O_1O}{O_2O} \cdot \frac{O_2Q_1}{PQ_1} \cdot \frac{PP_1}{O_1P_1} = 1$$

If r_1 and r_2 be the radii of the two circles, we have—

$$\frac{PP_1}{PQ_1} = \frac{O_2O}{O_1O} \cdot \frac{r_1}{r_2} = \text{constant},$$

which, therefore, by Newton's theorem, gives the locus of P as a Cartesian oval.

This method is very useful not only for constructing the curve by points, but it supplies a very simple method of constructing the tangent. Chasles remarked that the tangent to the curve at P and tangents to the circles at the corresponding points meet in one point. It can also be seen from this theorem that the Cartesian does not consist of a single oval, as stated by Descartes and others, but of two conjugate ovals having no finite common points.

Ex. 1. Prove that the locus of a point the tangents from which to two fixed circles are connected by a linear relation is a Cartesian.

Ex. 2. Rays of light emanating from a luminous point are refracted from a surface separating two transparent media of different densities, and converge to a point. Find the nature of the separating surface.

Ex. 3. The base angles of a variable triangle move on two fixed circles, while the two sides pass through the centres of the circles, and the base passes through a fixed point on the line joining the centres; prove that the locus of the vertex is a Cartesian.

216. Other forms of the Cartesian :

The fixed points being F_1 and F_2 and the variable point P, let

$$F_1P=r_1,\quad F_2P=r_2,\quad F_1F_2=c \text{ and } \angle PF_1F_2=\theta.$$

We have $\quad lF_1P\pm mF_2P=nc$ (say)

$$\therefore\quad l^2F_1P^2+n^2c^2-2nclF_1P=m^2F_2P^2 \qquad \ldots \quad (1)$$

But $\quad F_2P^2=F_1P^2+F_1F_2^2-2F_1P.F_1F_2.\cos\theta$

$$=r_1^2+c^2-2r_1c\cos\theta$$

$\therefore$ The equation (1) becomes—

$$l^2r_1^2+n^2c^2-2nclr_1=m^2(r_1^2+c^2-2r_1c\cos\theta)$$

i.e., $$r_1^2(l^2-m^2)-2cr_1(nl-m^2\cos\theta)$$
$$+c^2(n^2-m^2)=0 \quad \ldots \quad (2)$$

which may be written in the simpler form—

$$r_1^2-2\ a+b\cos\theta)r_1+k^2=0 \qquad \ldots \quad (3)$$

If $(x,\ y)$ be the co-ordinates of P referred to F_1 as origin, the Cartesian equation of the curve takes the form—

$$\{r_1^2(l^2-m^2)+2cm^2x+(n^2-m^2)c^2\}^2=4n^2c^2l^2r_1^2 \ \ldots \quad (4)$$

If now we put

$$V\equiv(l^2-m^2)r_1{}^2+2cm^2x+(n^2-m^2)c^2$$

and $$U\equiv x^2+y^2$$

The above equation may be written in the form

$$V^2=4k^2U$$

It follows then that the curve may be regarded as the envelope of

$$4k^2t^2+2tV+U=0 \qquad \ldots \quad (5)$$

where t is a variable parameter, which represents a system of circles having their centres on the axis of x.

217. SALMON'S METHOD :

The equation of the Cartesian may generally be brought to the form—

$$C^2 = k^3 L \qquad \dots \quad (1)$$

when C represents a circle and L a right line, k being a constant.

From this form of the equation it is evident that the intersections of C and k are cusps, the cuspidal tangents meeting in the centre of C, which is consequently the *triple focus* of the curve, and the line $L=0$ is a bitangent.

The curve can be regarded as the envelope of the variable circle—

$$\lambda^2 kL + 2\lambda C + k^2 = 0 \qquad \dots \quad (2)$$

the centre of which moves along a line perpendicular to L.

The equation (2) will represent a point-circle, if its discriminant vanishes, which, however, gives three values of λ, corresponding to which there are then three foci of the curve.

If A, B, C represent any three of the variable circles, the equation of the envelope can be written as—

$$l\sqrt{A} + m\sqrt{B} + n\sqrt{C} = 0$$

whence $l\rho + m\rho' + n\rho'' = 0$, where ρ, ρ', ρ'' represent the distances of any point from the three foci.

Since $k^2 = 0$ represents a circle of the system corresponding to the value $\lambda = 0$, we may write the equation in the form—

$$l\rho + m\rho' = nk.$$

For further information of the different forms and the properties of an oval of Descartes, the student is referred to the paper by S. Roberts in the Proceedings of the London Mathematical Society, Vol. 3 (1869-71), pp. 106-127.

218. CASEY'S METHOD:

Casey,* as a particular case of bicircular quartics, gave the following method of generating a Cartesian:

If a circle cuts a given circle orthogonally, while its centre moves along another given circle, its envelope is a Cartesian oval.

If in the equation (7) of § 169, we put $a=b$, the equation reduces to—

$$4a^2\{(x+f)^2+(y+g)^2\}=(x^2+y^2-c)^2 \qquad \ldots \quad (1)$$

which may be put into the form $C^2=k^3L$, and therefore represents a Cartesian oval.

The point $(-f, -g)$, *i.e.*, the centre of the fixed orthogonal circle is evidently a focus, and from what has been said in § 172, it follows that the limiting points of the two fixed circles are also foci of the curve. The Cartesian has therefore *three* foci which lie on a right line.

219. FOCI OF CARTESIANS:

Take O the middle point of F_1F_2 as origin and $F_1F_2=2a$, so that $F_1O=OF_2=a$

The equation of the curve can then be written as—

$$l\sqrt{(x-a)^2+y^2}+m\sqrt{(x+a)^2+y^2}=n \qquad \ldots \quad (1)$$

Now, transform this equation by the substitution—

$$x+iy=\alpha/\gamma \quad \text{and} \quad x-iy=\beta/\gamma$$

so that we obtain—

$$l\sqrt{(\alpha-a\gamma)(\beta-a\gamma)}+m\sqrt{(\alpha+a\gamma)(\beta+a\gamma)}=n\gamma$$

or, clearing the radicals this now takes the form:

$$\{l^2(\alpha-a\gamma)(\beta-a\gamma)-m^2(\alpha+a\gamma)(\beta+a\gamma)\}^2$$
$$-2n^2\gamma^2\{l^2(\alpha-a\gamma)(\beta-a\gamma)+m^2(\alpha+a\gamma)(\beta+a\gamma)\}$$
$$+n^4\gamma^4=0 \qquad \ldots \quad (2)$$

* Casey—Transactions of the Royal Irish Academy (1869).

This form of the equation shows that the two points ($\alpha=\gamma=0$ and $\beta=\gamma=0$), *i.e.*, the two circular points are double points on the curve.

The tangents at the point $\alpha=\gamma=0$ are given by—

$$\{l^2(\alpha-a\gamma)-m^2(\alpha+a\gamma)\}^2=0$$

It is therefore a cusp, the equation of the cuspidal tangent being—

$$\alpha=\frac{l^2+m^2}{l^2-m^2}\,a\gamma$$

which in Cartesian co-ordinates becomes—

$$x+iy=\frac{l^2+m^2}{l^2-m^2}\cdot a$$

Similarly, the other circular point is a cusp and the tangent at that point is—

$$x-iy=\frac{l^2+m^2}{l^2-m^2}\cdot a.$$

These two tangents evidently intersect at the real point—

$$x=\frac{l^2+m^2}{l^2-m^2}a, \quad y=0$$

which is therefore the *triple* focus of the curve.

In order to determine the ordinary foci, we consider the intersections of the line $\gamma=k\alpha$ with the curve, *i.e.*, we put $\gamma=k\alpha$ in equation (2), which becomes now divisible by α^2. We then determine k so that the remaining factor, regarded as an equation in α/γ has a double root.

Thus k is found to satisfy the equation—

$$(p^2-q^2)(kn^2+2ap)=0$$

where
$$p\equiv l^2(1-ak)-m^2(1+ak),$$
$$q\equiv l^2(1-ak)+m^2(1+ak).$$

This gives three values of k and the three tangents drawn to the curve from the circular point $\alpha=\gamma=0$ are finally given by the equations :

$$x+iy=a \qquad x+iy=-a,$$

$$x+iy=\frac{2a^2(l^2+m^2)-n^2}{2a(l^2-m^2)}.$$

Similarly, the three conjugate tangents drawn from the other circular point $\beta=\gamma=0$ are—

$$x-iy=a, \qquad x-iy=-a$$

and
$$x-iy=\frac{2a^2(l^2+m^2)-n^2}{2a(l^2-m^2)}.$$

These tangents intersect at the three points—

$$x=a, \quad y=0, \qquad x=-a, \quad y=0,$$

and
$$x=\frac{2a^2(l^2+m^2)-n^2}{2a(l^2-m^2)}, \quad y=0$$

which are the ordinary foci of the curve and it is evident that the three foci are collinear.

The first two are obviously the two fixed points F_1, F_2, and the third was first obtained by Chasles * from geometrical considerations. He shewed that a third point F_3 could be found on the line F_1F_2 whose distance ρ'' from the variable point P satisfies a relation of the form $l\rho \pm n\rho''=$constant, and the third point is a focus.†

* Chasles—*Aperçu historique*, Note XXI, p. 352.

† In fact, a bicuspidal quartic having two cusps at the circular points at infinity has three collinear ordinary foci. When these foci are real, the curve is that studied by Descartes; while when two are imaginary, the curve is still called a Cartesian, though Descartes' mode of generation is no longer applicable. For a detailed study of the curve, the student is referred to the authors cited above and to Sylvester—Phil. Mag., Vol. 31 (1866), Liguine—Bull. des Sciences Mathématiques—Second series, Vol. 6 (1882), p. 40, Intermédiare des Mathématiciens, Vol. 3 (1896), p. 238.

220. Equation referred to the triple focus:

From what has been said above, it is easily seen that the triple focus lies on the line on which the three ordinary foci lie.

If now the origin is taken at the triple focus, the distances x_1, x_2, x_3 of the three foci from the origin can easily be calculated from the values obtained in the preceding article.

Thus we find—

$$\alpha \equiv x_1 + x_2 + x_3 = -\frac{4a^2(l^2+m^2)+n^2}{2a(l^2-m^2)}$$

$$\beta \equiv x_1x_2 + x_2x_3 + x_3x_1 = \frac{4a^2l^2m^2+(l^2+m^2)n^2}{(l^2-m^2)^2}$$

$$\gamma \equiv x_1x_2x_3 = -\frac{2al^2m^2n^2}{(l^2-m^2)^3}$$

whence a new equation of the oval is obtained * in the form—

$$(x^2+y^2-\beta)^2+4\gamma(2x-\alpha)=0 \qquad \dots \quad (1)$$

This equation can be derived directly as follows:

Taking the line of the foci as the axis of x, the equation of a quartic with cusps at the circular points takes the form—

$$(x^2+y^2)^2+k(x^2+y^2)x+(ax^2+by^2+2gx+c)=0. \quad \dots \quad (2)$$

The curve is of class six and has one triple focus O and three real ordinary foci F_1, F_2, F_3.

If now O is taken as origin, the foci F_1, F_2, F_3 are the points $(x_1, 0)$, $(x_2, 0)$, $(x_3, 0)$.

The lines $y=\pm ix$ are asymptotes twice over, whence from equation (2) we must have—

$$k=0 \qquad \text{and} \qquad a=b$$

* See Panton—*The Educational Times*, Question 2622.

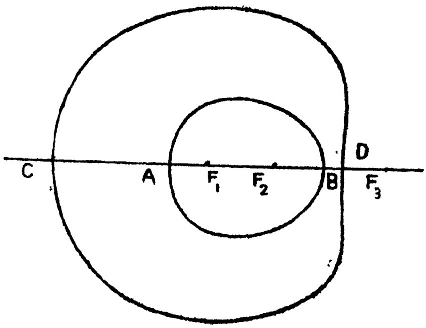
Again, the line $y=i(x-p)$ will be a tangent to (2),

if $$4gp^3+(4c-a^2)p^2-2agp-g^2=0$$

This is a cubic in p, whose roots are the distances of F_1, F_2, F_3 from the origin O, *i.e.*, x_1, x_2, x_3.

$$\therefore \quad \alpha=\frac{a^2-4c}{4g} \qquad \beta=-a/2 \qquad \gamma=g/4$$

whence $g=4\gamma$; $a=-2\beta$ and $c=\beta^2-4\alpha\gamma$.



$\therefore$ The equation (2) becomes—

$$(x^2+y^2)^2-2\beta(x^2+y^2)+8\gamma x+(\beta^2-4\alpha\gamma)=0$$

i.e. $$\{x^2+y^2-(x_1x_2+x_2x_3+x_3x_1)\}^2$$

$$+4x_1x_2x_3\{2x-(x_1+x_2+x_3)\}=0 \quad \ldots \quad (3)$$

This equation is clearly of the form $C^2=k^3L$ and therefore represents a Cartesian oval. (§ 218)

The equation is symmetrical in x_1, x_2, x_3 and consequently the curve stands in identical relations to the three foci F_1, F_2, F_3. It easily follows that the distances of

the foci from any point on the Cartesian are connected by a linear homogeneous relation of the form—

$$l\rho+m\rho'+n\rho''=0$$

as was otherwise stated in Article 217.

The equation (3) shows that the line $x=\frac{1}{2}(x_1+x_2+x_3)$ is the only bitangent to the curve, the points of contact lying on the circle—

$$x^2+y^2=x_1x_2+x_2x_3+x_3x_1$$

221. Position of the foci relative to the curve:

Let the curve be defined by the equation of the usual form $\rho\pm\mu\rho'=a$, where ρ, ρ' represent the distances of any point on the curve from the foci F_1, F_2, while μ and a are constants, $a > F_1F_2$, $1 > \mu > 0$.

It is easily seen that the curve consists of two ovals, one lying inside the other, the former corresponding to the equation $\rho+\mu\rho'=a$ and the latter to $\rho-\mu\rho'=a$.

Let the first oval cut the line F_1F_2 in A and B, where A lies on the left of F_1 (Figure of Art. 220). Then we have—

$$F_1A+\mu F_2A=a$$

$$\therefore \quad F_1A=a-\mu F_2A=a-\mu(F_2F_1+F_1A)$$

whence $F_1A=\dfrac{a-\mu c}{1+\mu}$, where $F_1F_2=c$.

Similarly, $F_2B=\dfrac{a-c}{1+\mu}$ and $AB=\dfrac{2a}{1+\mu}$.

Since $a>c$ and $\mu<1$, F_1A and F_2B are both positive, and consequently both the foci F_1, F_2 lie inside the first oval.

The two ovals do not intersect; for in that case, the curve, which is a quartic, would have four double points. This is also seen from the fact that the only common values of ρ and ρ' satisfying both the equations are $\rho=a$, $\rho'=0$, which is absurd; since when $\rho'=0$, $\rho=c$, and $c<a$.

Hence it follows that the first oval lies inside the second, and the two foci F_1, F_2 lie within both the ovals.

Again, let the second oval cut the line F_1F_2 in C and D, where C lies to the left of F_1.

Then, as before, from the second equation, we obtain—

$$F_1D = \frac{a-\mu c}{1-\mu}, \qquad F_1F_3 = \frac{a^2-\mu^2c^2}{c1(-\mu^2)}$$

$$F_1F_3 - F_1D = \frac{(a-\mu c)(a-c)}{c(1-\mu^2)}$$

$\therefore$ $F_1F_3 > F_1D$, and consequently F_3 lies outside both the ovals.

It is to be noted that each oval is the inverse of the other with respect to the internal focus F_1, and either oval turned through two right angles is the inverse of the other with regard to the central focus F_2, while each oval is its own inverse with respect to the external focus F_3. The two tangents drawn from F_3 to the two ovals respectively are equal. The two ovals possessing these properties are called *conjugate* ovals.

We have discussed here the case when the three foci are all real, for the case when two of them are imaginary and for further information the student is referred to Cayley's *Memoir on Caustics*—Coll. Papers, Vol. 2, p. 336.

Ex. 1. Prove that in a Cartesian the rectangle under the segments, made by the inner oval on any transversal from the external focus, is constant.

Ex. 2. Prove that the Cartesian—

$$r^2 - 2(a + b\cos\theta)r + c^2 = 0$$

has three real foci, or only one, according as $a-b >$ or $\angle\, c$.

Ex. 3. If a radius vector be drawn from the focus F_1 cutting two conjugate ovals in P and Q, show that the tangents at P and Q intersect at the middle point of the arc PQ of the circle passing through P, Q and the other two foci F_2, F_3. (See Basset—*Cubics and Quartics*, § 268.)

222. THE INVERSE OF A CARTESIAN :

The equation of a Cartesian referred to its triple focus can be written as $C^2 = k^3 L$, the centre of the circle being the triple focus, and the line L being the bitangent. Now, the focus of a curve is inverted into a focus of the inverse curve, and when the circular points are cusps, the origin is a focus. Hence the inverse of a Cartesian with respect to any point is a bicircular quartic having three foci on a circle passing through the origin, which is also a focus.

Ex. 1. Prove that two Cartesian curves with the same real ordinary foci cut orthogonally.

[Invert the curves *w. r. t.* one of their intersections and we obtain two circular cubics with four concyclic foci and their real asymptotes perpendicular.]

Ex. 2. Inverting the curves in *Ex.* 1, prove that any two bicircular quartics with the same four real concyclic foci cut orthogonally.

Ex. 3. Prove that through any point can be drawn two Cartesian ovals having three given points in direction for foci, and the two curves cut orthogonally.

223. THE TANGENT AND THE NORMAL AT ANY POINT :

The normal at any point of a Cartesian can be constructed by the usual method applicable to all curves in bi-polar co-ordinates.

In the case of the oval corresponding to the equation $\rho + \mu\rho' = a$, on the radii PF_1 and PF_2 measure $PA = 1$ and $PB = \mu$, and complete the parallelogram PANB. Then the diagonal PN will be the required normal at P.

The equation of the normal at the point $P(x, y)$ is—

$$\frac{X - x}{\dfrac{\partial\rho}{\partial x} + \mu\dfrac{\partial\rho'}{\partial y}} = \frac{Y - y}{\dfrac{\partial\rho}{\partial y} + \mu\dfrac{\partial\rho'}{\partial y}}$$

But from the equation of the curve, we easily obtain

$$\frac{\partial r}{\partial x}=\frac{x}{\rho}=\cos\omega, \qquad \frac{\partial \rho'}{\partial x}=\frac{x-c}{\rho'}=\cos\omega'$$

$$\frac{\partial r}{\partial y}=\frac{y}{\rho}=\sin\omega, \qquad \frac{\partial \rho'}{\partial y}=\frac{y}{\rho'}=\sin\omega'$$

where $\omega=\angle \mathrm{PF}_1 x, \qquad \omega'=\angle \mathrm{PF}_2 x.$

$\therefore$ The equation of the normal becomes—

$$\frac{\mathrm{X}-x}{\cos\omega+\mu\cos\omega'}=\frac{\mathrm{Y}-y}{\sin\omega+\mu\sin\omega'}$$

The co-ordinates of N are given by—

$$x_1=x-(\cos\omega+\mu\cos\omega') \quad \text{and} \quad y_1=y-(\sin\omega+\mu\sin\omega')$$

when N is found to be the fourth vertex of the parallelogram.

A second construction is given as follows:

Describe a circle through P and the two foci F_1, F_2, and let Q be the second point in which F_1P meets this circle. Then the line joining P to R, the middle point of the arc cut off by PQ is the normal at P.

This construction depends upon a theorem given by Prof. Crofton.*

Ex. 1. Prove that the normal at any point of a Cartesian $m\rho+n\rho=c$ divides the angle between the focal radii into parts whose sines are in the ratio $m:n$.

Ex. 2. Show that the caustic by refraction of a circle is the evolute of a Cartesian (Salmon—*H. P. Curves*, § 116, p. 100).

Ex. 3. Prove that the tangent at P bisects the angle between the focal distance F_1P and the tangent at P to the circle F_2PF_1.

* Crofton—*Transactions, London Mathematical Society* (1866).

224. THEOREM:

If any chord meet a Cartesian in four points, the sum of their distances from any focus is constant.

The polar equation of the curve, referred to any focus as pole, may be written as—

$$r^2-2r(a+b\cos\theta)+c^2=0 \qquad \dots \quad (1)$$

Let $$r(l\cos\theta+m\sin\theta)=1 \qquad \dots \quad (2)$$

be the polar equation of a chord.

If now we eliminate θ between these equations, we obtain a biquadratic in r, in which the co-efficient of r^3 is $-4a$, and consequently, the sum of the roots, which represent the distances of the points of intersection from the pole, is constant, and this proves the proposition.

225. POINTS OF INFLEXION:

Since a Cartesian is a bicuspidal quartic, Plücker's formula gives at once that a Cartesian has eight points of inflexion.

We shall now prove that *these eight points of inflexion lie on a circular cubic.**

Let the equation of a Cartesian be written in the form:

$$(r^2-k^2)^2+A_1x+B_1=0 \qquad \dots \quad (1)$$

Since the curvature at a point of inflexion vanishes and changes sign, the radius of curvature becomes infinite at a point of inflexion. Hence the denominator in the expression for the radius of curvature, when equated to zero, gives the equation of the curve passing through the inflexions.

* S. Roberts—*On the Ovals of Descartes*—Proc. of the London Math. Soc., Vol. 3 (1869-71), pp. 106-127.

Differentiating the equation (1) twice in succession, we obtain—

$$4(r^2-k^2)(x+yp)+A_1=0 \quad \dots \quad (2)$$

and
$$8(x+yp)^2+4(r^2-k^2)(1+p^2+yp_1)=0 \quad \dots \quad (3)$$

where
$$p\equiv(\partial y/\partial x) \quad \text{and} \quad p_1\equiv(\partial p/\partial x)$$

$\therefore$ For a point of inflexion we obtain

$$8(x+yp)^2+4(r^2-k^2)(1+p^2)=0 \quad \dots \quad (3')$$

Eliminating p between (2), (3′), and simplifying by means of (1), we obtain the locus of inflexions in the form

$$\{16k^2(A_1x+B_1)-A_1^2\}(r^2-k^2)$$
$$-8(A_1x+B_1)(A_1x+2B_1)-2A_1^2y^2=0$$

which evidently represents a circular cubic.

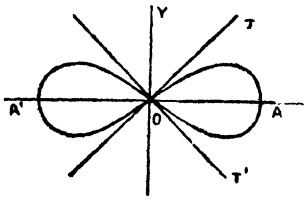
226. The Lemniscate of Bernoulli:

We have already seen (§ 208) that when $a=c$, the Cassinian curve reduces to a lemniscate of Bernoulli,* and its equation then becomes—

$$(x^2+y^2)^2-2a^2(x^2-y^2)=0 \quad \dots \quad (1)$$

This equation can again, by a simple transformation of the co-ordinates, be put into the form—

$$(x^2+y^2)^2=4a^2xy \quad \dots \quad (2)$$



* For the historical introduction, the student is referred to G. Loria—*Sp. Alg. und trans. ebene Kurven*, Vol. I (1910), p. 215.

The polar equation of the curve becomes—

$$r^2 = 2a^2 \cos 2\theta \qquad \ldots \quad (3)$$

The shape of the curve is that of the figure of eight, and the equation (1) shows that the origin is a double point, with a pair of perpendicular tangents bisecting the angles between the axes. These tangents are also inflexional tangents to the curve. Hence the origin is a biflecnode. Since a Cassinian has two biflecnodes at the circular points, we obtain the theorem :

The lemniscate of Bernoulli is a curve of order four with three biflecnodes, two of which lie at the circular points at infinity.

It is easy to see that the lemniscate is the inverse of a rectangular hyperbola with respect to its centre.

The curve is also the pedal of a rectangular hyperbola with respect to its centre.

For, if the line $X \cos \theta + Y \sin \theta = p$ is to be a tangent to the hyperbola $x^2 - y^2 = a^2$, it must be identical with the equation $x.X - y.Y = a^2 Z$, whence we obtain—

$$a^2 (\cos^2 \theta - \sin^2 \theta) = p^2.$$

$\therefore$ The polar equation of the pedal is—

$$r^2 = a^2 \cos 2\theta$$

which is a lemniscate of Bernoulli.

Ex. 1. Prove that the envelope of circles described on the central radii of an ellipse as diameters is a lemniscate.

Ex. 2. Show that the locus of inflexions of a family of Cassinian curves with given singular foci is a lemniscate of Bernoulli.

Ex. 3 The vertices of an equilateral hyperbola move along the circumference of a circle. Prove that the locus of the foci of all rectangular hyperbolas which circumscribe the triangle and have a given centre is a lemniscate of Bernoulli (Basset—*loc. cit.*, § 255)

Ex. 4. Show that $\{y^2 + (a+x)^2\}\{y^2 + (a-x)^2\} = a^4$ is a lemniscate of Bernoulli.

227. THE NORMAL:

Let the normal at any point (x, y) on the lemniscate represented by the equation (1) of the preceding article make an angle ϕ with the x-axis.

$$\text{Then,} \quad \tan\phi = -\frac{\partial c}{\partial y} = -\frac{a^2+(x^2+y^2)x}{a^2-(x^2+y^2)y}$$

$$= -\frac{(1+2\cos 2\theta)}{(1-2\cos 2\theta)} \cdot \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin 3\theta}{\cos 3\theta} = \tan 3\theta$$

$$\therefore \quad \phi = 3\theta.$$

i.e., the angle made with the x-*axis by the normal at any point is three times the angle made by the radius vector to the point.*

This gives a very simple method of constructing the normal and consequently the tangent at any point of the lemniscate. The problem of the trisection of an angle can very easily be solved, simply by drawing the normal and the tangent in a given direction at any point of the lemniscate. Thus the points, the normals at which are parallel to the bisector of the angle between the positive axes, are given by $\theta = \pi/12$, $9\pi/12$, $12\pi/12$. Similarly, the points where the normals are parallel to the y-axis have $\theta = \pm\pi/6$.

228. PEDAL FORM AND RADIUS OF CURVATURE:

Let ON be the perpendicular drawn from the centre O on the tangent at any point on the lemniscate

$$r^2 = a^2 \cos 2\theta \qquad \ldots \quad (1)$$

If ϕ be the angle OPN, we have—

$$\tan \text{PON} = \cot\phi = -\frac{\partial r}{r\partial\theta}$$

From equation (1) of the curve, we obtain—

$$2 \log r = 2 \log a + \log (\cos 2\theta)$$

Differentiating this with respect to θ, we obtain—

$$\frac{\partial r}{r\partial\theta} = -\tan 2\theta = -\tan \mathrm{PON}$$

$$\therefore \quad \angle \mathrm{PON} = 2\theta$$

But, $\quad p = \mathrm{ON} = \mathrm{OP} \cos \mathrm{PON} = r \cos 2\theta = \dfrac{r^3}{a^2}$

whence we obtain $\qquad r^3 = a^2 p \qquad \ldots \quad (2)$

which is the pedal equation of the curve.

The equation of the reciprocal polar curve is—

$$r^{\frac{2}{3}} \cos \tfrac{2}{3}\theta = a^{\frac{2}{3}}$$

and the tangential equation of the curve is—

$$27a^4(\lambda^3 + \mu^2)^2 = \{4 - a^2(\lambda^2 - \mu^2\}^3 \qquad \ldots \quad (3)$$

showing that the curve is of class six. The radius of curvature is obtained from equation (2) in the form

$$\rho = r\frac{\partial r}{\partial p} = \frac{a^2}{3r}.$$

229. Parametric form :

Since the lemniscate is a curve of order four possessing three nodes, it is certainly a unicursal curve; and we can express the co-ordinates of any point on the curve as rational functions of a parameter λ in the form—

$$x = a\sqrt{2}.\ \frac{\lambda + \lambda^3}{1 + \lambda^4}, \qquad y = a\sqrt{2}\ \frac{\lambda - \lambda^3}{1 + \lambda^4} \qquad \ldots \quad (1)$$

If we replace λ by $1/\lambda$, it is seen that the value of x remains unchanged, while the value of y changes sign. Therefore the values λ and $1/\lambda$ give two points which are

symmetrical with respect to the x-axis. Points symmetrical with respect to the centre correspond to two equal but opposite values of λ. If the diameter makes an angle ϕ with the axis, we have—

$$\tan\phi=\frac{1+\lambda^2}{1-\lambda^2}$$

$$\therefore \qquad \lambda^2=\frac{1-\tan\phi}{1+\tan\phi} \qquad \ldots \quad (2)$$

giving two equal but opposite values of λ.

The equation of the line joining any two points (λ) and (μ) is obtained in the form—

$$(1+\lambda\mu)\{(\lambda+\mu)^2-(1+\lambda^2\mu^2)\}x$$
$$+(1-\lambda\mu)\{(\lambda+\mu)^2+(1+\lambda^2\mu^2)\}y-2\sqrt{2}\,a\lambda\mu(\lambda+\mu)=0 \quad (3)$$

whence the equation of the tangent at any point λ is—

$$(1+\lambda^2)(4\lambda^2-1-\lambda^4)x$$
$$+(1-\lambda^2)(4\lambda^2+1+\lambda^4)y-4\sqrt{2}\,a\lambda^3=0 \quad \ldots \quad (4)$$

If 'm' be the slope of this tangent, we have—

$$-m=\frac{(1+\lambda^2)(4\lambda^2-1-\lambda^4)}{(1-\lambda^2)(4\lambda^2+1+\lambda^4)}=\text{constant}$$

This being of degree six in λ, the curve is of class six and six tangents can be drawn parallel to any given direction.

From equation (1), it can be written as—

$$\frac{1-3\tan^2\phi}{3\tan\phi-\tan^3\phi}=-m=\text{constant}$$

i.e., $$\tan 3\phi=\text{constant}.$$

$\therefore$ If ϕ_1 be one root, the other two roots are—

$$\phi_1+\frac{\pi}{3}, \qquad \phi_1+\frac{2\pi}{3}$$

Thus we obtain the theorem :

There are six tangents to a lemniscate parallel to a given direction, the points of contact lying on diameters mutually inclined at an angle $\pi/3$.

The condition of collinearity of three points takes the form—

$$\lambda_2\lambda_3+1/\lambda_2\lambda_3+\lambda_3\lambda_1+1/\lambda_3\lambda_1+\lambda_1\lambda_2+1/\lambda_1\lambda_2=0$$

and four points are concyclic, if—

$$\lambda_1.\lambda_2.\lambda_3.\lambda_4=1.$$

If, therefore, the osculating circle at a point (λ) meets the curve at (μ), we must have—

$$\lambda^3\mu=1$$

the point (μ) is called the *satellite* of the point (λ).

The satellites of four concyclic points $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are then connected by a similar relation, whence we may say that *the satellites of four concyclic points are concyclic.*

The equation says that for a given value of μ there are three values of λ, namely, λ_1, λ_2, λ_3, and we have—

$$\lambda_1+\lambda_2+\lambda_3=0, \quad \lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2=0, \quad \lambda_1\lambda_2\lambda_3=1/\mu$$

and these points lie on the line—

$$(1+\mu^2)x+(1-\mu^2)y-2\sqrt{2}\,\mu=0$$

which is a line perpendicular to the radius vector through the point. Thus we obtain the theorem :

Through each point on a lemniscate there pass three circles osculating the curve elsewhere, and the points of osculation lie on a line perpendicular to the radius vector to the point.

Again, when the curve is defined in bi-polar co-ordinates, by the equation $\rho\rho' = a^2$, we may put this in the parametric form—

$$\rho = ae^u = a(\cosh u + \sinh u)$$

$$\rho' = ae^{-u} = a(\cosh u - \sinh u)$$

where u is a parameter.

If PD be the bisector of the angle F_1PF_2, we have—

$$F_1D + F_2D = 2a$$

and
$$\frac{F_1D}{F_2D} = \frac{\rho}{\rho'}.$$

Consequently, we must have—

$$F_1D = a(1 + \tanh u), \qquad F_2D = a(1 - \tanh u)$$

and
$$OD = a \tanh u.$$

But from elementary geometry, we obtain—

$$F_1P \cdot F_2P = PD^2 + F_1D \cdot F_2D$$

i.e.
$$\rho\rho' = a^2 = PD^2 + a^2(1 - \tanh^2 u)$$

whence
$$PD = a \tanh u = OD.$$

Ex. 1. Show that the quartic

$$xyz^2 = (x^2 + y^2)^2$$

can be projected into the lemniscate of Bernoulli

$$r^2 = a^2 \sin 2\theta.$$

Ex. 2. P is any point on a lemniscate with F_1 and F_2 as foci and O as centre. If PD be the bisector of the $\angle F_1PF_2$, prove that PD = OD.

Ex. 3. Prove that the points of contact of the four tangents drawn to a lemniscate from any point of it are collinear.

Ex. 4. Show that the biflecnode at the origin is a double focus of a lemniscate of Bernoulli.

230. THE LIMAÇON :

The polar equation of a Cartesian oval is—

$$r^2-2(a+b\cos\theta)r+k^2=0$$

When $k=0$, the equation reduces to—

$$r=a+b\cos\theta \qquad \dots \quad (1)$$

which represents a curve having, in addition to the two cusps at the circular points, the origin for a node and the curve is called a *Limaçon.* The curve was first studied by Pascal, who so named it from a fancied resemblance to the form of a nail.

The equation (1) shows that the Limaçon is the inverse of a central conic with respect to a focus.*

The polar equation of a conic, the focus being the pole, may be written as $l/r=1+e\cos\theta$

and that of its inverse is —

$$r=a+b\cos\theta, \qquad \dots \quad (2)$$

consequently, the curve is the inverse of an ellipse when $a>b$ and is called an elliptic limaçon. When $a<b$, it is the inverse of a hyperbola and is called a Hyperbolic limaçon. When, however, $a=b$, the curve is called a Cardioid from its heart-like shape, and is the inverse of a parabola with respect to its focus. The different species of the curve are shown in the diagram of § 234.

When $a=2b$, the curve is called the Trisectrix, a curve by means of which any given angle can be readily trisected.

* C. Taylor—*Ancient and Modern Geometry of Conics*—Cambridge (1881), p. 356.

231. THE CARTESIAN EQUATION:

The Cartesian equation of a limaçon is obtained in the form:

$$(x^2+y^2-bx)^2=a^2(x^2+y^2) \qquad \dots \quad (1)$$

The tangents at the origin are

$$\sqrt{b^2-a^2}\;\; x \pm ay=0$$

and the origin is an acnode or a crunode, according as $a >$ or, $< b$, *i.e.*, according as it is elliptic or hyperbolic. If $a=b$, the origin is a cusp. It is a curve of order four, having the circular points as cusps; the equations of the cuspidal tangents being—

$$x \pm iy + b/2=0$$

Since the limaçon is a quartic with two cusps and a double point, it is of the fourth class, and accordingly possesses one bitangent and two inflexions. It is a unicursal curve and its reciprocal is of the same species.

If $a<b$, the curvature of the curve at O is given by—

$$1/\sqrt{b^2-a^2}.$$

Writing the equation in the form—

$$(x^2+y^2+bx-\tfrac{1}{2}a^2)^2-(\tfrac{1}{4}a^4-a^2bx)=0$$

we see that the bitangent of the limaçon is the line

$$4bx-a^2=0$$

the ordinates of the points of contact being given by

$$4by=\pm a\sqrt{4b^2-a^2}$$

These points of contact are real or imaginary, according as $a <$ or $> 2b$, *i.e.*, according as the curve is an elliptic or hyperbolic limaçon.

When $a=2b$, the two points of contact coincide, and we have an *undulation* at the point where the x-axis meets the curve, *i.e.*, at the vertex A. (Fig. Art. 237.)

Ex. 1. On the radius vector to a fixed circle from a fixed point on it, a portion of fixed length is taken on either side of the circle. Show that the locus of the extremities of the length is a limaçon.

[The polar equation is $r=p\cos\omega\pm k$]

Ex. 2. Prove that the caustic by reflexion of a circle is the evolute of the limaçon.

[The equation of the caustic is of the form $r=p(1+e\cos\omega)$. See Theory of Plane Curves, Vol. I, § 131.]

Ex. 3. Prove that the locus of the vertex of an angle of given magnitude, whose sides touch two given circles is composed of two limaçons.

Ex. 4. If the legs of a given angle slide on two given circles, prove that the locus of any carried point is a limaçon and the envelope of any carried line is a circle.

[See Williamson's Diff. Calc., Chap. *XIX*.]

Ex. 5. Show that a limaçon may be generated as an epitrochoid * by the rolling of one circle upo another of equal radius.

232. Points of inflexion:

Since the limaçon has a node at the origin and a pair of imaginary cusps at the circular points, the curve belongs to the ninth species, and consequently there are only two points of inflexion.

In order to determine the inflexions, we obtain from the polar equation

$$r=a+b\cos\theta$$

the two following parametric forms:

$$\left.\begin{aligned} x&=b\cos^2\theta+a\cos\theta \\ \text{and}\qquad y&=b\cos\theta\sin\theta+a\sin\theta \end{aligned}\right\} \quad \cdots \quad (1)$$

or

$$\left.\begin{aligned} 2x&=b+b\cos 2\theta+2a\cos\theta \\ 2y&=b\sin 2\theta+2a\sin\theta \end{aligned}\right\} \quad \cdots \quad (2)$$

* These curves belong to a general class called *Roulettes* which will be discussed in the next chapter.

The condition of collinearity of three points (α), (β), (γ) is

$$\begin{vmatrix} b+b\cos 2\alpha+2a\cos\alpha & b\sin 2\alpha+2a\sin\alpha & 1 \\ b+b\cos 2\beta+2a\cos\beta & b\sin 2\beta+2a\sin\beta & 1 \\ b+b\cos 2\gamma+2a\cos\gamma & b\sin 2\gamma+2a\sin\gamma & 1 \end{vmatrix}=0$$

which, on simplification and after rejecting a factor, reduces to

$$4b^2\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2}+ab\{4\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}$$
$$+\cos\alpha+\cos\beta+\cos\gamma-\cos(\alpha+\beta+\gamma)\}+2a^2=0.$$

If now we put $\alpha=\beta=\gamma=\lambda$, we obtain—

$$4b^2+ab(4\cos^3\lambda+3\cos\lambda-\cos 3\lambda)+2a^2=0$$

i.e.
$$2b^2+3ab\cos\lambda+a^2=0$$

The vectorial angle of the point of inflexion is then given by—

$$\cos\lambda=-\frac{2b^2+a^2}{3ab} \qquad \dots \quad (3)$$

whence the existence of two inflexions is further manifest. It is further seen that these are real, if $b<a<2b$. If the limaçon has a node, they are imaginary; but they always lie on the real line—

$$x=\frac{(4b^2+2a^2)(b^2-a^2)}{9a^2b}$$

which is obtained by combining (3) with the equation of the curve.

The inflexions may further be obtained from the condition that the radius of curvature is infinite, and consequently—

$$r^2-r\frac{\partial^2 r}{\partial\theta^2}+2\left(\frac{\partial r}{\partial\theta}\right)^2=0$$

whence $\cos\lambda=-\dfrac{2b^2+a^2}{3ab}$, $r_\lambda=\dfrac{-2(b^2-a^2)}{3a}$

where r denotes the corresponding radius vector.

233. PARAMETRIC FORM :

Since the limaçon is a unicursal curve, the co-ordinates of its points can be expressed in terms of a parameter.

The equations—

$$x=b\cos^2\theta+a\cos\theta, \qquad y=b\cos\theta\sin\theta+a\sin\theta$$

evidently show that the co-ordinates are expressed in terms of the parameter θ. We may obtain an other form, if we put $t=\tan\theta/2$

when $\quad \sin\theta=2t/(1+t^2), \quad \cos\theta=(1-t^2)/(1+t^2)$

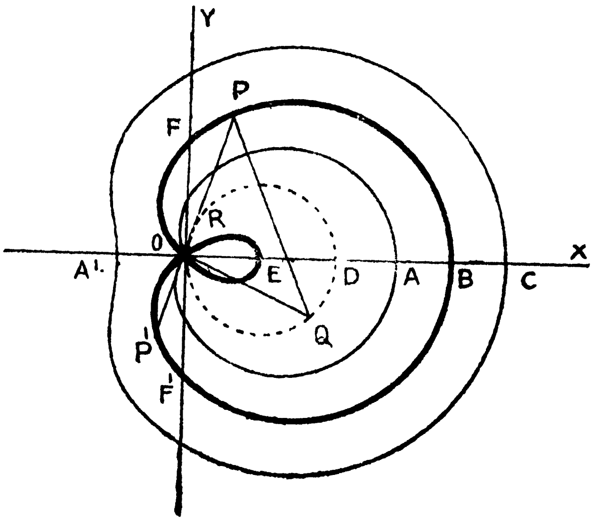
$$\therefore \qquad x=\frac{1-t^2}{(1+t^2)^2}\{(a+b)+(a-b)t^2\}$$

and
$$y=\frac{2t}{(1+t^2)^2}\{(a+b)+(a-b)t^2\}$$

234. CONSTRUCTION OF THE LIMAÇON :

The limaçon can be generated as a locus or as an envelope in a number of ways.

Let OQD be a circle with centre E and diameter



OD$(=b)$. Let a straight rod PP′ of given length $(2a)$ move in such a manner that its middle point R always lies on the

given circle, while the rod always passes through a fixed point O on the circumference of the circle. Then, the locus of the end points P and P′ is a limaçon. The limaçon can thus be constructed mechanically.

Taking OD as the initial line, let (r, θ) be the co-ordinates of P.

Then. $r = \text{OP} = \text{RP} + \text{OR} = a + b \cos\theta$ ... (1)

Similarly, $r' = \text{OP}' = a - b \cos\theta$,

which is the same equation as (1), when θ is increased by π.

Hence the locus of P and P′ is the same curve defined by the equation (1) which represents a limaçon The form of the curve is shown by the dark line in the diagram.

The tangent and the normal at any point can be easily constructed by Chasle's method.*

Through O draw OQ perpendicular to the radius vector OP to the point, meeting the fixed circle in Q. Join PQ; then PQ is the normal at P.

For rectification and other particulars, the student is referred to F. G. Teixeira—*loc. cit.*, pp. 152-153.

235. A SECOND METHOD:

The Limaçon is the pedal of a circle with respect to any point in its plane.

Let C be the centre of a circle and O any point in its plane.

At any point A on the circle draw a tangent, and OP the perpendicular on the tangent.

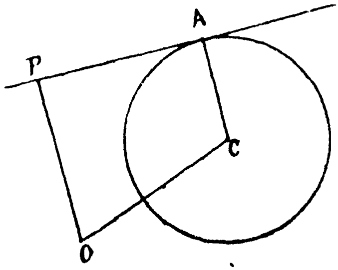
Let a be the radius of the circle,

$$\text{OC} = b \quad \text{and} \quad \angle \text{ACO} = \pi - \theta.$$

* Chasles—*Histoire de la Geometrie*, pp. 548-9.

From the geometry of the figure, we obtain

$$OP = AC + OC \cos \angle POC = a + b \cos \theta.$$



Therefore, the locus of P is given by $r = a + b \cos \theta$, which is an elliptic or hyperbolic limaçon, according as $a >$ or $< b$, *i.e.*, according as O lies within or without the circle. When O lies on the circle, the locus is a Cardioid.

The limaçon can readily be traced by drawing from a fixed point on a circle any number of chords, and taking off a constant length on each of these chords, measured from the circumference of the circle.

236. THE PEDAL EQUATION:

The polar equation of the curve is $r = a + b \cos \theta$ ... (1)

If ϕ be the angle between the tangent at any point P and the radius vector, then $p = r \sin \phi$

and $$\cot \phi = \frac{\partial r}{r \partial \theta} = -\frac{b \sin \theta}{r}$$

i.e. $$r \cot \phi = -b \sin \theta.$$

$$\therefore \quad \frac{r^2}{p^2} = \operatorname{cosec}^2 \phi = 1 + \cot^2 \phi = 1 + \frac{b^2 \sin^2 \theta}{r^2} \quad \ldots \quad (2)$$

Eliminating θ between (1) and (2), we obtain—

$$\frac{r^2 - p^2}{p^2} = \frac{b^2 - (r-a)^2}{r^2}$$

i.e. $$r^4 = p^2 (b^2 - a^2 + 2ar),$$

which is the required equation.

237. THE FOCI OF THE LIMAÇONS:

In order to find the singular focus we must determine the tangents at the circular points. For this we transform the Cartesian equation (1) of § 231 into trilinear co-ordinates by putting—

$$x+iy=a/\gamma \quad \text{and} \quad x-iy=\beta/\gamma$$

when the transformed equation becomes—

$$\{2a\beta-b(a+\beta)\gamma\}^2=4a^2a\beta\gamma^2 \qquad \dots \quad (1)$$

Therefore the tangents at the two circular points $(\beta=\gamma=0)$ and $(a=\gamma=0)$ are respectively—

$$2\beta-b\gamma=0 \quad \text{and} \quad 2a-b\gamma=0,$$

which in Cartesian co-ordinates become—

$$2(x-iy)-b=0 \quad \text{and} \quad 2(x+iy)-b=0$$

These two intersect at the point $x=\frac{1}{2}b$ and $y=0$.

$\therefore$ The point $S(b/2, 0)$ is the *triple* focus.

To determine the ordinary foci, we have to find the tangents drawn from the circular points, *i.e.*, find the condition that the line $a=k\gamma$ meets the curve (1) in two coincident points.

Putting $a=k\gamma$ in (1) we must obtain an equation in β and γ, which is to be a perfect square. The condition for this will determine the value of k, for which $a=k\gamma$ will be a tangent. Similarly, the tangent drawn from the other circular point may be obtained.

These two tangents are found to intersect in the point

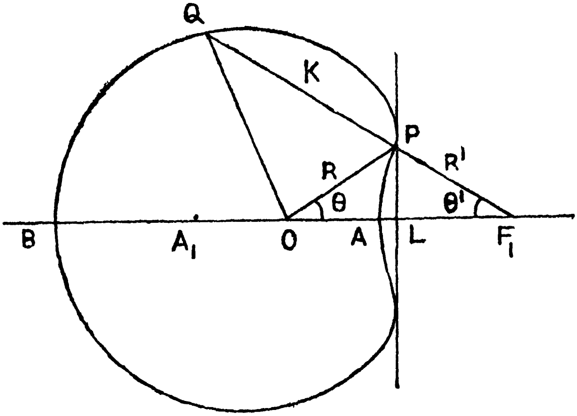
$$F_1\left(-\frac{a^2-b^2}{2b},\ 0\right)$$

which is an ordinary focus of the curve.

If the vectorial angles are measured from the initial line OB, which meets the curve in B and A, we have

$$OA=a-b, \qquad OB=a+b$$

$$\therefore \quad AB=2a$$



If now A_1 be taken on the opposite side of OA,

$$OS=\tfrac{1}{4}A_1B.$$

But $$OF_1=\frac{a^2-b^2}{2b}$$

$$\therefore \quad AF_1=OF_1-OA=\frac{a^2-b^2}{2b}-(a-b)$$

$$=(a-b)^2/2b.$$

Consequently, AF_1 is positive and the point F_1 lies outside the curve.

Ex. 1. If P is any point on the limaçon with O as node and F_1 as external focus, show that

$$\angle OF_1P = 2\angle OPT$$

where PT is the tangent at P.

[Inverting *w.r.t.* the node we obtain a conic with the tangent equally inclined to the focal distances of the point of contact.]

Ex. 2. Prove that a limaçon is self-inverse *w.r.t.* the circle through the node with its centre at the ordinary focus.

[The equation referred to the ordinary focus is—

$$4b^2r^2 - 4br(b^2 + a^2 \cos\theta) + (a^2 - b^2)^2 = 0$$

which is unchanged by replacing r by $(a^2 - b^2)^2/4b^2r$]

Ex. 3. Show that the locus of inflexions of the family

$$r = a + b\cos\theta$$

when b is kept constant and a varies, is—

$$r^2 + br\cos\theta + 2b^2\sin^2\theta = 0.$$

Ex. 4. Show that the points of contact of bitangents in Ex. 3 lie on the line

$$r + b\cos\theta = 0.$$

Ex. 5. Show that the distances of any point on a limaçon from the node and the ordinary focus are rational functions of the co-ordinates of the point.

238. The centre of inversion:

The bi-polar equation of the curve, referred to O and F_1, is obtained in the form—

$$2ar + 2br' = a^2 - b^2 \qquad \dots \quad (1)$$

where $$r' = F_1P.$$

The Cartesian equation of the curve may be obtained from this by substituting the values of r and r'.

Let $\angle PF_1O = \theta'$, $\quad OF_1 = s = \frac{1}{2}(a^2 - b^2)/b$

The polar equation of the curve, referred to F_1, is obtained in the form:

$$r'^2 - r'(b^2 + a^2 \cos\theta')/b + s^2 = 0 \qquad (2)$$

If now F_1PQ be any chord, we have—

$$F_1P + F_1Q = (b^2 + a^2 \cos\theta')/b \qquad \cdots \quad (3)$$

and

$$F_1P . F_1Q = s^2 \qquad \cdots \quad (4)$$

If K be the middle point of PQ, we have—

$$2F_1K = F_1P + F_1Q = (b^2 + a^2 \cos\theta')/b$$

whence the locus of the mid-point K is the hyperbolic limaçon—

$$r = \tfrac{1}{2}(b^2 + a^2 \cos\theta')/b$$

From equation (4) it is seen that the curve is its own inverse with respect to the focus F_1 as origin, which is consequently a centre of inversion.

Ex. 1. Prove that triangles OF_1P and F_1QO are similar and the circum-circle of the triangle OPQ touches the line OF_1 at O.

Ex. 2. Show that the angles made by the tangents at P and Q with F_1P are equal.

[The curve is self-inverse *w.r.t.* F_1.]

Ex. 3. Show that the points of contact of the bitangent are given by $\cos\theta = -\frac{1}{2}a/b$ and $r = \frac{1}{2}a$.

Ex. 4 If T be the middle point of the arc PQ of the circle circumscribing OPQ, show that TP and TQ are the tangents at P and Q.

[Using the bi-polar equation and differentiating *w.r.t.* s, we obtain $a\cos\phi = -b\cos\phi'$, where ϕ and ϕ' are the angles which the tangent makes with the radii.]

Ex. 5. Show that $\quad r(1-k^2) = a(\cos\theta - k)$,

where k is a parameter, represents a family of limaçons with a common node and focus,

239. THEOREM:

The locus of the point of intersection of two tangents at the extremities of a chord through the node is a nodal circular cubic.

The equation of the limaçon being of the form—

$$(x^2+y^2-bx)^2=a^2(x^2+y^2) \qquad \ldots \quad (1)$$

if we find the equation of the first polar (polar cubic) of any point (x', y'), and transforming this to polar co-ordinates, eliminate r between this and the polar equation of the curve, we shall obtain—

$$\{(a^2+bx')\tan^2\theta-2by'\tan\theta+a^2+b^2-bx'\}^2$$
$$=a^2(y'\tan\theta-2b+x')^2(1+\tan^2\theta) \qquad \ldots \quad (2)$$

Equation (2) then determines the vectorial angles of the points of contact of the four tangents drawn to the curve from the point (x', y').

Writing this in the form—

$$(a^2+bx')\sin^2\theta-2by'\sin\theta\cos\theta+(a^2+b^2-bx')\cos^2\theta$$
$$=a\{y'\sin\theta+(x'-2b)\cos\theta\} \qquad \ldots \quad (3)$$

we see that if PP′ be the extremities of the chord POP′, their vectorial angles are respectively θ and $\theta+\pi$, and if P, P′ be the points of contact of two tangents drawn from (x', y'), the equation (3) must be satisfied by θ and $\pi+\theta$.

This requires that both sides of (3) must vanish,

i.e., $(a^2+bx')\sin^2\theta-2by'\sin\theta\cos\theta+(a^2+b^2-bx')\cos^2\theta=0$

and $\qquad y'\sin\theta+(x'-2b)\cos\theta=0$

whence eliminating θ, we obtain—

$$(a^2+bx')(x'-2b)^2+2by'^2(x'-2b)+k^2(a^2+b^2-bx')=0$$

i.e. $\qquad (a^2+bx')(x'-2b)^2+y'^2(a^2-3b^2+bx')=0$

Transforming the origin to the point $(2b, 0)$, the locus of (x', y') is found to be the curve—

$$bx(x^2+y^2)+(a^2+2b^2)x^2+(a^2-b^2)y^2=0 \quad \ldots \quad (4)$$

which is evidently a nodal circular cubic.

240. THE CARDIOID : *

We have already seen that when $a=b$, the limaçon becomes a Cardioid. The Cartesian equation of the curve can therefore be written in the form—

$$(x^2+y^2)^2-2ax(x^2+y^2)=a^2y^2 \qquad \ldots \quad (1)$$

showing that the origin is a cusp.

The polar equation may be written in the form—

$$r=a(1+\cos\theta) \qquad \ldots \quad (2)$$

It is consequently a tricuspidal quartic and is the inverse of a parabola with respect to its focus. It is also the pedal of a circle with respect to a point on its circumference. It is easy to see from equation (2) that the length of a chord through the origin (which is a cusp) is constant and equal to $2a$.

Since the origin is a cusp, it is the limiting position of the three single foci of an oval of Descartes. Hence the cusp is the triple focus.

241. PARAMETRIC FORM :

The Cardioid is a tricuspidal quartic and consequently a unicursal curve. The co-ordinates of any point can be expressed as—

$$x=\frac{2a(1-\lambda^2)}{(1+\lambda^2)^2}, \qquad y=\frac{4a\lambda}{(1+\lambda^2)^2} \qquad \ldots \quad (1)$$

where λ is a parameter.

From Plücker's formula it is easily seen that the curve has no inflexion. The equation of the bitangent is—

$$x+a/4=0$$

* Properties of this curve were discussed by Raymond Clare Archibald in his inaugural Dissertation—*The Cardioide and some of its related curves*—(Strassburg 1906).

The equation of the tangent at any point (λ) is obtained in the form—

$$(3\lambda^2-1)x+\lambda(\lambda^2-3)y+2a=0 \quad \dots \quad (2)$$

Since this involves λ in the third degree, for a given value of (x, y), there are three values of λ, and consequently the curve is of class three.

The tangential equation of the curve, in Boothian coordinates, is obtained in the form—

$$27a^4(\xi^2+\eta^2)-2a^2(a\xi+2)^3=0$$

The tangent (2) meets the bitangent at the point

$$\left(-\frac{a}{4}, \frac{3a}{4\lambda}\right)$$

The line joining this to the point $(a/2, 0)$, which is the singular focus, is given by—

$$x+\lambda y=a/2$$

If this line makes an angle ω with the x-axis, we have—

$$\tan\omega=-1/\lambda. \quad \dots \quad (3)$$

Now, let the tangent (2) make an angle α with the x-axis, so that we have—

$$\tan\alpha=-\frac{3\lambda^2-1}{\lambda(\lambda^2-3)} \quad \dots \quad (4)$$

If λ be eliminated between (3) and (4), we obtain—

$$\tan\alpha=\frac{3\tan\omega-\tan^3\omega}{1-3\tan^2\omega}=\tan 3\omega,$$

whence the three values of ω are respectively—

$$\frac{\alpha}{3}, \quad \frac{\alpha}{3}+\frac{\pi}{3}, \quad \frac{\alpha}{3}+\frac{2\pi}{3}.$$

Thus, the points in which three parallel tangents to a Cardioid meet the bitangent are joined to the point $(a/2, 0)$ by three lines making angles

$$\frac{a}{3}, \quad \frac{a}{3}+\frac{\pi}{3}, \quad \frac{a}{3}+\frac{2\pi}{3}$$

with the x-axis.

We thus obtain the theorem :

Three parallel tangents to a Cardioid meet the bitangent in a triad of points which are seen from the singular focus at angles $\pi/3$ apart.

242. Collinear points :

The parameters of the four points in which any line

$$lx+my+1=0$$

intersects the Cardioid are determined by the equation—

$$\lambda^4+2(1-al)\lambda^2+4am\lambda+2al+1=0 \quad \ldots \quad (1)$$

which is obtained by substituting in the equation of the line the values of x and y in terms of λ.

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the roots of this equation, we have—

$$\Sigma\lambda_1=0, \qquad \Sigma\lambda_1\lambda_2=2(1-al), \qquad \Sigma\lambda_1\lambda_2\lambda_3=-4am$$

$$\lambda_1\lambda_2\lambda_3\lambda_4=1+2al.$$

Eliminating λ_4 and l between these, we obtain the condition of collinearity of any three points in the form—

$$\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)+\lambda_1^2+\lambda_2^2+\lambda_3^2$$
$$+\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1+3=0 \quad \ldots \quad (2)$$

The parameters of the points of contact of the three tangents drawn from any point (x', y') are found from the equation of the tangent (2) of the last article to satisfy the equation—

$$\lambda^3+\frac{3x'}{y'}\lambda^2-3\lambda+\frac{2a-x'}{y'}=0 \quad \ldots \quad (3)$$

If we divide the left-hand side of (1) by that of (3), and denote the quotient by Q and the remainder by R, we obtain—

$$Q=\lambda-3\frac{x'}{y'},$$

$$R=\left(5-2al+9\frac{x'^2}{y'^2}\right)\lambda^2+\left(4am-8\frac{x'}{y'}-\frac{2a}{y'}\right)\lambda$$

$$+\left(2al+1+3\frac{x'}{y'}\cdot\frac{2a-x'}{y'}\right)$$

The conditions that the three points of contact of the tangents drawn from the point (x', y') lie on the line (1) are

$$5-2al+9\frac{x'^2}{y'^2}=0, \qquad 4am-8\frac{x'}{y'}-\frac{2a}{y'}=0$$

and

$$2al+1+\frac{3x'}{y'}\cdot\frac{2a-x'}{y'}=0$$

The first two determine the parameters l, m of the line, and by eliminating l from the third, we obtain—

$$x'^2+y'^2+ax'=0 \qquad \ldots \quad (4)$$

which, when (x', y') are regarded as variables, represents a circle whose centre is the point $(-a/2, 0)$ and radius $a/2$. Hence we obtain the theorem:

The points of contact of the three tangents drawn from a point on the circle with centre $(-a/2, 0)$ *and radius* $a/2$ *lie on a right line.*

The fourth point where the line through the points of contact meets the curve again is given by the parameter—

$$\lambda=3\frac{x'}{y'}.$$

243. THEOREM:

The tangents at the extremities of a chord through the origin (cusp) intersect at right angles on a fixed circle whose centre is the triple focus.

From equation (3) we obtain

$$\lambda_1+\lambda_2+\lambda_3=-3x'/y', \qquad \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1=-3,$$

and

$$\lambda_1\lambda_2\lambda_3=-(2a-x')/y' \qquad \dots \quad (5)$$

Substituting these values in (2) we obtain the locus of (x', y') in the form—

$$x'^2+y'^2+ax'=0$$

as was otherwise obtained in the preceding article.

The equation of the tangent at any point (μ) is—

$$(3\mu^2-1)x+\mu(\mu^2-3)y+2a=0 \qquad \dots \quad (6)$$

If this is perpendicular to the tangent at the point (λ), we must have—

$$(\lambda\mu+1)\{\lambda^2\mu^2-3(\lambda^2+\mu^2)+8\lambda\mu+1\}=0$$

which shows that the orthoptic locus of a Cardioid is a degenerate curve, in fact, a sextic consisting of a circle and a quartic.

If now the tangents at λ_1 and λ_2 are perpendicular, we may take $\lambda_1\lambda_2=-1$, and consequently

$$\lambda_3=(2a-x')/y', \qquad \lambda_1+\lambda_2=-2(a+x')/y'.$$

Substituting these values in the second of the equations (5), we obtain the locus of (x', y') as—

$$x^2+y^2-ax=2a^2$$

which is a circle with centre $(a/2, 0)$ and radius $\frac{3}{2}a$. The point $(a/2, 0)$ is the triple focus, and it is easily seen that the chord joining the points λ_1 and λ_2 passes through the origin, which proves the proposition.

244. Reciprocal polar of the Cardioid :

Let A be the vertex, F the cusp and FY the perpendicular on the tangent PT at any point P on the Cardioid

$$r=a(1+\cos\theta) \qquad \ldots \quad (1)$$

The equation (1) may be written as—

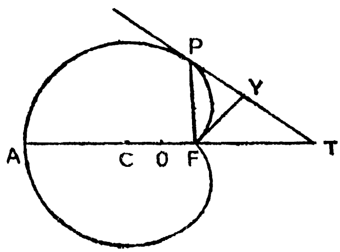
$$r=2a\cos^2\frac{\theta}{2}, \qquad i.e., \quad r^{\frac{1}{2}}=(2a)^{\frac{1}{2}}\cos\frac{\theta}{2}.$$

whence the pedal equation may be written as—

$$r^{\frac{1}{2}+1}=(2a)^{\frac{1}{2}}p, \quad \text{or,} \quad r^3=2ap^2 \qquad \ldots \quad (2)$$

and if $\angle AFY=\omega$, the equation of the pedal or the tangential polar equation * is—

$$p=2a\cos^3\tfrac{1}{3}\omega \qquad \ldots \quad (3)$$



The reciprocal polar is the curve given by—

$$r^{\frac{1}{3}}\cos\tfrac{1}{3}\theta=(2a)^{\frac{1}{3}}, \quad \text{or,} \quad r\cos^3\tfrac{1}{3}\theta=2a \qquad \ldots \quad (4)$$

The chord of curvature is given by

$$\gamma=p\frac{\partial r}{\partial p}=\tfrac{2}{3}r.$$

Ex. Show that

$$r=a(1-\cos\theta) \quad \text{and} \quad r=b(1+\cos\theta)$$

represent two systems of curves cutting at right angles for all values of a and b.

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* Williamson—Diff. Calc., § 190.

245. THEOREM:

The envelope of the directrix of a parabola, touching a given circle and having its focus at a fixed point on the circle, is a Cardioid.

Let C be the centre of the fixed circle of radius a and F the focus of the parabola touching the circle at P. From F draw the perpendicular FX on the directrix.

If now, $\angle XFP=\theta$, $\angle XFC=\psi$ and also $FC=a$

then $\angle XFC-\angle XFP=\psi-\theta=CFP=CPF=\frac{1}{2}\theta$ *

$$\therefore \quad 2\psi=3\theta$$

Also, $FX=FP+FP\cos\theta=FP(1+\cos\theta)$

$$=2a\cos\frac{\theta}{2}\,.\,2\cos^2\frac{\theta}{2}$$

$$=4a.\cos^3\tfrac{1}{2}\theta$$

$$=4a\cos^3\tfrac{1}{3}\psi$$

which is the tangential polar equation of the Cardioid, whence the proposition is proved.

Ex. 1. Prove that the evolute of a Cardioid is another Cardioid.

Ex. 2. Show that the Cardioid is the locus of a point on the circumference of a circle which rolls on the inside of another circle of half its diameter.

Ex. 3. Prove that the entire length of the Cardioid is eight times the diameter of the generating circle.

Ex. 4. The orthoptic locus of a Cardioid is a circle and a limaçon.

Ex. 5. Show that the quartic, with one real and two unreal cusps,

$$(x^2+y^2-2xz)^2=4z^2(x^2+y^2)$$

can be projected into a Cardioid by a real projection.

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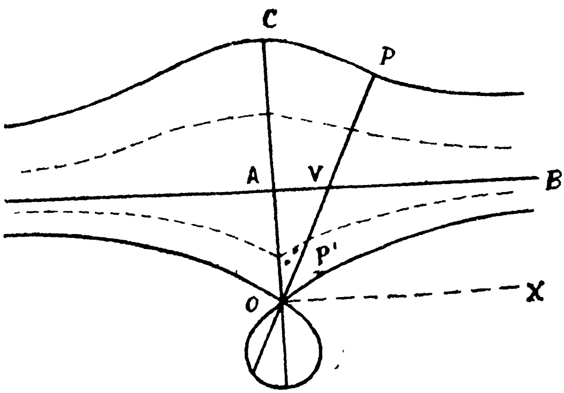
* Salmon's Conics, § 217.

246. THE CONCHOID OF NICOMEDES:

Nicomedes, the Greek Geometer, for the purpose of trisecting an angle, invented an interesting curve of order four, which is called the Conchoid or Cochloid. The curve has been defined as follows:

If through any fixed point O, a secant $Op'p$ be drawn meeting a fixed right line AB in V, and on this two points p and p' be taken such that $Vp = Vp' =$ a constant length l, then the locus of p or p' is called the Conchoid.

Draw OA perpendicular upon AB, and let $OA = a$.



Now taking O for pole and the line OX, parallel to AB, for the initial line, the polar equation of the locus is—

$$r = a \operatorname{cosec} \theta \pm l \qquad \dots \quad (1)$$

The curve consists of two branches, having the line AB for a common asymptote.

In the above equation, the + sign refers to the branch more remote from AB, and the − sign refers to the branch nearer to AB. These two branches are called the *superior* and the *inferior* branch respectively.

The Cartesian equation of the curve is—

$$(x^2 + y^2)(y - a)^2 = l^2 y^2$$

which includes both the branches.

The origin is a double point, the tangents being given by

$$a^2x^2+(a^2-l^2)y^2=0 \qquad \dots \quad (2)$$

It is, therefore, a node, a cusp, or a conjugate point, according as a is $<$, $=$, or $> l$; *i.e.*, OA is $<$, $=$, or $> \sqrt{p}$.

The conchoid is a curve of order four, which passes through the circular points. It has a real *tacnode* at infinity, with AB as the tacnodal tangent; in fact, the base AB is an asymptote to the two branches.

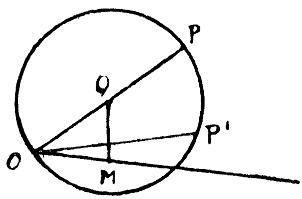
A circle which passes through the double point cannot intersect the curve in more than four points. When $l > a$, the curvature of both the branches at the origin is given by

$$a/l\sqrt{l^2-a^2}$$

The curve was used for trisection of an angle, and the insertion of two mean proportionals between two given straight lines.

247. Trisection of an angle :

We shall now show how the conchoid can be employed for trisecting a given angle POM.



Let P be any point on OP, and from Q, the middle point of OP, draw QM ($=a$), perpendicular on OM, and let $OQ=l$.

Through P draw the Conchoid $r=a \operatorname{cosec}\theta+l$, O being the origin and OM the initial line. With Q as centre and

OQ as radius, describe a circle, cutting the other branch of the conchoid in P′. Then OP′ trisects the angle POM.

Let $\angle$ POM$=\theta$ and $\angle$ P′OM$=\phi$.

Then, OP′$=a$ cosec $\phi-l$, since P′ lies on the lower branch of the curve. But QM$=a=l\sin\theta$,

and $$\text{OP}'=\text{OP}\cos\text{POP}'=2l\cos(\theta-\phi)$$

$$\therefore\quad 2l\cos(\theta-\phi)=a\operatorname{cosec}\phi-l$$

i.e., $2l\cos(\theta-\phi)\sin\phi=a-l\sin\phi=l\sin\theta-l\sin\phi$.

$$\therefore\quad 2\cos(\theta-\phi)\sin\phi=\sin\theta-\sin\phi$$

or, $$\sin(\theta-2\phi)=\sin\phi$$

$$\therefore\quad \theta=3\phi.$$

Consequently, OP′ trisects the angle POM.

The trisection of an angle can also be effected by means of the *Trisectrix*. (§ 94)

248. Points of inflexion :

In order to determine the inflexions we consider the formula—

$$r^2-r\frac{\partial^2 r}{\partial\theta^2}+2\left(\frac{\partial r}{\partial\theta}\right)^2=0 \qquad \ldots \quad (3)$$

Substituting from the equation of the curve, we obtain the equation

$$l\sin^3\theta\pm 3a\sin^2\theta\mp 2a=0 \qquad \ldots \quad (4)$$

Now, putting $\sin\theta=1/t$, the equation becomes—

$$\pm 2at^3\mp 3at-l=0$$

which gives three values of t, which are all real, if $a>l$; but one of them is to be rejected, since the roots must not lie between $+1$ and -1. If $a<l$, only one root is real.

Hence a Conchoid with a node has two real inflexions, with an acnode four and with a cusp only two inflexions.

From the equation (1) we obtain for the ordinate of an inflexion $\sin\theta=\frac{y-a}{l}$, and substituting this in equation (4) we obtain—

$$y^3-3a^2y+2a(a^2-l^2)=0 \quad \ldots \quad (5)$$

This shows that the ordinates of the inflexions are determined by an equation of the third degree, which, however, reduces to a quadratic, if we put $l^2=2a^2$

i.e., $$(y+a)(y^2-ay-2a^2)=0 \text{ *}$$

249. PARAMETRIC FORM:

Since the Conchoid of Nicomedes is a unicursal curve, the co-ordinates of its points may be expressed in terms of a parameter. From equation (1) we may write—

$$x=a\cot\theta+l\cos\theta \quad y=a+l\sin\theta.$$

Again, we may write—

$$-(y+l-a)=(y-l-a)t^2$$

$$\therefore \quad y=\frac{a-l+(a+l)t^2}{1+t^2}$$

and $$x=-\frac{2t[a-l+(a+l)t^2]}{t^4-1}$$

* A geometrical construction for the inflexions was given by Huygens—*Œuvres de Huygens*—Vol. 2, pp. 245-46. For complete discussion the student is referred to G. Loria, *loc. cit.*, pp. 140-141. See also F. G. Teixeira, *loc. cit.*, pp. 188-190.

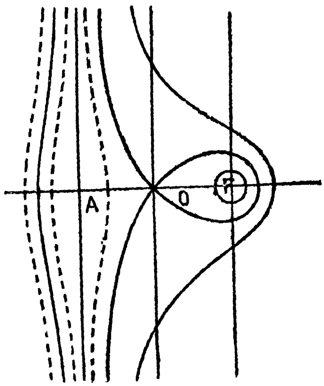
250. THE CONCHALE : *

We shall now consider a class of curves which may be defined in the following manner:

Let F be a fixed point and r a fixed straight line. Then the locus of a point P such that the product of its distances from F and r is constant, is a curve studied by O. Schlömilch, which was, however, given by Huber many years ago. This is called a Conchale in virtue of the affinity it has with the Conchoid of Nicomedes.

From F draw FA perpendicular on the fixed line r. Let O, the middle point of FA, be the origin and the line FA as the axis of x. Let $FA=2a$.

If (x, y) be the co-ordinates of the point P, and ak be the constant product, we have—



$$(x+a)^2\{(x-a)^2+y^2\}=a^2k^2 \qquad \dots \quad (1)$$

which is the Cartesian equation of the Conchale.

The equation shows that it is a curve of order four symmetrical with respect to the x-axis. The abscissae of the four points in which it is met by the x-axis are

* A systematic investigation of the properties of this curve has been given by G. Huber—Die Konchalen, ihre orthogonalen Trajectorien und die Cissoiden vierter Ordnung—Monatshefte Math. Phys., Vol. 6 (1895).

$\pm\sqrt{a(a\pm k)}$, while the y-axis meets it in points at distances $\pm\sqrt{k^2-a^2}$ from the origin.

Of the four points, four or two are real, according as k is $<$ or $>a$, while both are real or imaginary, according as k is $<$ or $>$ a.

When $k=a$, the equation reduces to

$$y^2(x+a)^2+x^2(x^2-2a^2)=0 \qquad \dots \quad (2)$$

The origin is a node, with the tangents $y=\pm x\sqrt{2}$, and it intersects the x-axis in points $x=\pm a\sqrt{2}$.

In the general case the curve passes through the circular points, and has a tacnode at infinity on the axis of y, the tacnodal tangent being $x+a=0$, which is the fixed line r. The different forms of the curve corresponding to the values of the constant a and k are exhibited in the figure.

When $k<a$, the curve consists of two infinite branches and an oval.

When $k=a$, it consists of a *serpentine* and a nodal branch.

When $k>a$, it consists of two infinite branches.

The tangent at any point P of a Conchale can be constructed as a special case of a general method given by A. Hurwitz * as follows:

Join PF and at F draw p perpendicular to PF. Then join P with the point Q where p meets the given line r and draw a line through Q which is the harmonic conjugate of this line *w.r.t.* p and r. This line will then be parallel to the tangent required.

For other particulars of this curve, the student should consult G. Loria, *loc. cit.*, pp. 204-208.

* A Hurwitz—*Über Tangentenkonstruktionen*—Math. Ann. Bd. 22 (1883).

251. ORTHOGONAL TRAJECTORIES:

If k is regarded as a parameter, the equation (1) represents a singly infinite system of curves; the differential equation of their orthogonal trajectories is—

$$dy[2x(x-a)+y^2]-y(x+a)dx=0.$$

Integrating we obtain—

$$\frac{y^4-4axy^2}{\{y^2-2a(x-a)\}^2}=\text{constant}$$

whence the trajectories are rational curves of order four with F as a common double point and a tacnode at infinity on the axis of x, the line at infinity being the corresponding tacnodal tangent.

252. THE VIRTUAL PARABOLA:

A special quartic curve, belonging to the class of curves known as symmetric polyzomal curves of order four, was originally studied by G. Saint Vincent,* and is called the *virtual parabola*. This was subsequently discussed, under different forms and names, by other workers, such as Cramer,† Magnus,‡ Schlömich,§ Huygens,|| etc., and as many as six methods of constructing the curve were given by them.

The curve studied by Vincent is defined by the equation—

$$(x^2-by)^2=a^2(x^2-y^2) \qquad \dots \quad (1)$$

* G. Saint Vincent—*Opus geometricum quadraturae*, etc. (1647).

† Cramer—*Introduction à l' Analyse des Lignes courbes* (1750), p. 451.

‡ Magnus—*Sammlung von Aufgaben und Lehrsätzen, etc.* (Berlin) 1833, p. 286.

§ Schlömich—*Übungsbuch zum Studium der höheren Analysis.* (Leipzig) 1878, p. 87.

|| Huygens—*Œuvres de Huygens*—II (1889), p. 70.

The equation shows that the axis of y is the axis of symmetry of the curve. Solving the equation for y we obtain—

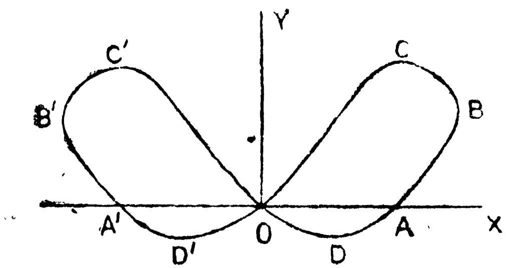
$$y = \frac{bx^2 \pm ax\sqrt{a^2+b^2-x^2}}{a^2+b^2} \quad \ldots \quad (2)$$

whence $$\frac{\partial y}{\partial x} = \frac{2bx}{a^2+b^2} \pm \frac{a(a^2+b^2-2x^2)}{(a^2+b^2)\sqrt{a^2+b^2-x^2}}$$

Thus the origin is a double point and the tangents are given by the equation—

$$y = \pm \frac{ax}{\sqrt{a^2+b^2}}$$

If x lies between 0 and $\sqrt{a^2+b^2}$, to each value of x correspond two real values of y; one is positive and the other negative in the interval $x=0$, $x=a$, when $y=0$; while both are positive in the interval $x=a$ to $x=\sqrt{a^2+b^2}$



If $x > \sqrt{a^2+b^2}$, y is imaginary and at the point B, the tangent is parallel to the y-axis. The co-ordinates of B are $(\sqrt{a^2+b^2}, b)$.

The points of maximum and minimum ordinates are determined by putting $\frac{\partial y}{\partial x}=0$.

But from the equation of the curve we obtain—

$$\frac{\partial y}{\partial x}=\frac{-2x^3+a^2x+2bxy}{(a^2+b^2)y-bx^2}=0$$

$$\therefore \quad 2by+a^2-2x^2=0.$$

Eliminating x between this and the equation of the curve, we obtain—

$$y=\frac{b\pm\sqrt{a^2+b^2}}{2}$$

which give the maximum and minimum values of the ordinates. The abscissae of the four points are given by—

$$x=\pm\frac{\sqrt{2}}{2}\sqrt{(a^2+b^2)\pm b\sqrt{a^2+b^2}}.$$

The equation (2) shows that the parabola given by—

$$y=\frac{b}{a^2+b^2}x^2$$

bisects the chords of the virtual parabola parallel to the axis of x.

The virtual parabola is a unicursal curve; the coordinates of its points can therefore be expressed in terms of a parameter.

Writing

$$x=\sqrt{a^2+b^2}.\ \frac{t^2-1}{t^2+1},$$

from equation (2) we obtain—

$$y=\frac{b(t^2-1)^2\pm 2at(t^2-1)}{(t^2+1)^2}.$$

253. Construction of the virtual parabola:

The following construction of the curve was given by Cramer.

Let OL be a variable chord of a fixed circle through a fixed point O on its circumference. Take O to be the origin. Through L draw LN parallel to the axis of x meeting the axis of y in N, and on this make NP equal to OL. Then the locus of P, as OL varies, is a *virtual parabola*.

Let us take the equation of the circle in the form

$$X^2+Y^2-aX-bY=0$$

and let (x, y) be the co-ordinates of P.

Now $\quad OL^2=X^2+Y^2=NP^2=x^2 \quad$ and $\quad y=Y$.

Substituting these values of X and Y in the equation of the circle, we obtain—

$$x^2-by=aX=a\sqrt{x^2-y^2}$$

i.e., $$(x^2-by)^2=a^2(x^2-y^2)$$

which represents a virtual parabola.

If in the equation of the curve we put $b=0$, it reduces to—

$$a^2y^2=x^2(a^2-x^2)$$

which is a special virtual parabola, called the *Lemniscate of Gerono* by Gabriel Marie.

G. Saint-Vincent gave a number of propositions in Vol. X of his work, describing the several methods of constructing the curve, for a detailed account of which the student is referred to G. Loria—*loc. cit.*, pp. 186-191.

254. Points of inflexion:

At a point of inflexion $\dfrac{\partial^2 y}{\partial x^2}=0$, and from the equation of the curve we obtain—

$$\frac{\partial^2 y}{\partial x^2}=\frac{2b}{a^2+b^2}\pm\frac{a x}{a^2+b^2}\cdot\frac{3(a^2+b^2)-2x^2}{(a^2+b^2-x^2)^{\frac{3}{2}}}$$

whence the abscissae of the points of inflexion are given by—

$$4b^2(a^2+b^2-x^2)^3=a^2x^2\{3(a^2+b^2)-2x^2\}$$

This is an equation of the third order in x^2, and gives three values of x^2 and six values of x, two and two equal but of opposite signs. Corresponding to each value of x, we obtain from equation (2) two values of y—one referring to the point of inflexi and the other to the second point on the curve having the same abscissa as the first.

255. The Lemniscate of Gerono:

If in the equation of the virtual parabola, we put $b=0$, the equation reduces to—

$$x^4=a^2(x^2-y^2) \qquad \ldots \quad (1)$$

which, therefore, represents a special virtual parabola, and is known as the Lemniscate of Gerono.* This curve has sometimes been confounded with Bernoulli's Lemniscate, owing to its form being that of a figure of eight. The origin is a biflecnode, and the axis of x meets it in the two points $(\pm a, 0)$.

* Gabriel Marie—*Exercises de Géométrie Descriptive.*

Differentiating twice the equation (1) we obtain—

$$\frac{\partial y}{\partial x}=\frac{a^2-2x^2}{a\sqrt{a^2-x^2}},\quad \text{and}\quad \frac{\partial^2 y}{\partial x^2}=\frac{x(3a^2-2x^2)}{a(a^2-x^2)^{\frac{3}{2}}}$$

$\therefore$ When $x=\pm a$, $\dfrac{\partial y}{\partial x}=\infty$; and if $x=\pm\dfrac{a}{\sqrt{2}}$,

$$\frac{\partial y}{\partial x}=0 \quad \text{and} \quad y=\pm\frac{a}{2}.$$

Also, when $x=0$, $\dfrac{\partial^2 y}{\partial x^2}=0.$

Therefore, the tangents at the points $(\pm a,\ 0)$ are perpendicular to the axis of x and at the points

$$\left(\pm\frac{a}{\sqrt{2}},\ \pm\frac{a}{2}\right)$$

the ordinate is a maximum and the tangents are parallel to the axis of x. In fact, the lines $y=\pm\dfrac{a}{2}$ are bitangents to the curve. The origin is a biflecnode, the tangents making an angle of 45° with the x-axis.

The curve has four imaginary inflexions at the points whose abscissae are $\pm a\sqrt{\frac{3}{2}}$, and it has a *tacnode* at infinity.

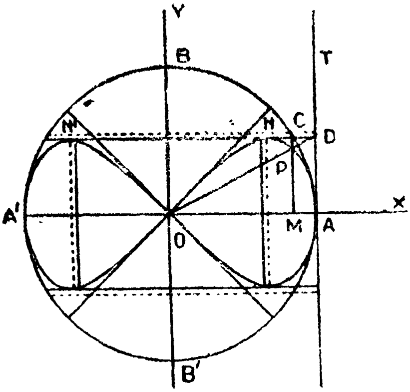
For, if we transform the equation (1) to trilinears by substituting $\dfrac{\gamma}{\beta}$ and $\dfrac{\alpha}{\beta}$ for x and y respectively, the transformed equation becomes—

$$\gamma^4=a^2(\gamma^2-\alpha^2)\beta^2 \qquad \dots \quad (2)$$

which represents a quartic with a tacnode at the point $\gamma=0$, $\beta=0$, with the line $\beta=0$ as the tacnodal tangent, *i.e.*, there is a tacnode at infinity on the axis of x.

256. Construction of the lemniscate:

Let O be the origin and centre of a circle of radius $OA=a$, and AT the tangent at A; and on AT take a point D such that $AD < OA$. Through D draw DN parallel to OA, meeting the circle at C.



Draw the ordinate $CM=DA$, meeting OD in $P(x, y)$.

Then the locus of P will be a lemniscate of Gerono.

From the triangles OMP and OAD, we have—

$$\frac{OM}{PM}=\frac{OA}{CM}$$

But $\quad CM^2=a^2-x^2 \qquad$ *i.e.*, $\quad CM=\sqrt{a^2-x^2}$

$$\therefore \quad \frac{x}{y}=\frac{a}{\sqrt{a^2-x^2}}$$

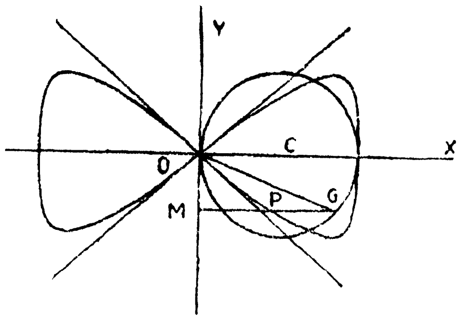
whence $\qquad x^2(a^2-x^2)=a^2y^2$

i.e. $\qquad x^4=a^2(x^2-y^2)$

which is the equation of a lemniscate of Gerono.

257. A SECOND METHOD:

Let C be the centre of a circle of diameter $OB = a$. Through O draw a chord OG and through G draw GM parallel to OB, meeting OY, the tangent at O, in M. Now on GM take a point P such that $MP = GB$. Then the locus of P is a lemniscate of Gerono.*



Taking OB and OY as the axes of x and y, let (x, y) be the co-ordinates of P and $\angle GOM = \omega$.

$$x = MP = BG = a\cos\omega \quad \text{and} \quad y = OM = OG\cos\omega$$
$$= a\sin\omega\cos\omega$$

Eliminating ω we obtain $x^4 = a^2(x^2 - y^2)$, which is a lemniscate of Gerono.

This may again be written in the form—

$$x = \sqrt{\frac{a^2}{4} + \frac{ay}{2}} + \sqrt{\frac{a^2}{4} - \frac{ay}{2}}.$$

This curve has been discussed by Cramer in his *Introduction*.

There are various other curves of the fourth order possessing interesting properties and forms, for which the student should consult Teixeira's book cited before.

* G. Saint-Vincent—*Opus geometricum quadraturae circuli a sectionum coni* (Antwerpiae 1647) Bd. 2, p. 482.

CHAPTER XII

TRANSCENDENTAL CURVES

258. ROULETTES :

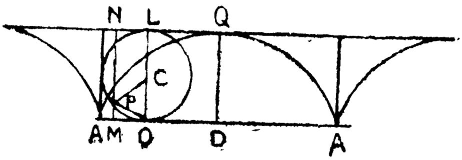
We have so far discussed the properties of curves whose equations are reducible to a finite number of terms involving positive integral powers of x and y. In this chapter we shall study the properties of curves represented by transcendental equations, which involve functions expressible by an infinite series of algebraic terms. All transcendental curves may be considered as curves of infinite degree. Any straight line cuts it in an infinite number of points, and it possesses an infinity of multiple points and multiple tangents. We shall consider, however, only the principal properties of some of the most remarkable of them. The most remarkable of the transcendental curves are those known as the *Roulettes*. When one curve rolls, without sliding, upon another, any point invariably connected with the rolling curve describes another curve called a *roulette*. The rolling curve is called the *generating curve* and the fixed curve is called the *directing curve* or the *base*. For a complete discussion of these curves, the student should consult Dr. Besant's *Notes on Roulettes and Glissettes* (1890).

259. THE CYCLOID :

The simplest of these curves is the *Cycloid*, which holds the first place among transcendental curves, both for historical interest and for the variety of its physical applications.

DEFINITION : The Cycloid is the path, traced out by a point on the circumference of a circle which rolls along a right line.

Let LPO be any position of the rolling circle, O being the point of contact with the fixed line. Take AO equal to the arc PO. Then A is the position of the generating point when in contact with the fixed line. If AA′ be equal to the circumference of the circle, A′ is the position of the generating point at the end of one complete revolution of the circle.



Draw DQ perpendicular to AA′ at its middle point D, and let $DQ=2a=$diameter of the circle.

Then evidently Q is the highest point in the cycloid.

Draw PM perpendicular to AA′, and let (x, y) be the co-ordinates of P referred to rectangular axes through A, so that $x=AM$, $y=PM$.

Denoting the angle PCO by θ, where C is the centre of the circle, we have—

$$\left.\begin{aligned} x&=AM=AO-MO=a(\theta-\sin\theta) \\ y&=PM=CO-PC\ \cos\theta=a(1-\cos\theta) \end{aligned}\right\} \quad \ldots \quad (1)$$

Then θ evidently represents the angle through which the circle has rolled, starting from the position when the moving point lies on the fixed line.

Hence the position of a point is determined when the angle θ is known.

Eliminating θ between the two relations (1), we obtain the equation of the curve in the form—

$$a-y=a\cos\left\{\frac{x+\sqrt{(2ay-y^2)}}{a}\right\} \quad \ldots \quad (2)$$

It is, however, more convenient to express the equation in the form (1).

The point Q is called the *vertex*, and it is sometimes convenient to express the equation referred to the tangent and normal at the vertex Q as axes of co-ordinates, and then the equations become—

$$x = QN = a(\theta' + \sin\theta'), \quad y = PN = a(1 - \cos\theta') \quad \dots \quad (3)$$

where $\theta' = \angle PCL = \pi - \theta$

260. THE TANGENT AND NORMAL TO CYCLOID:

It can be easily seen that at any instant of the motion of the generating circle, the lowest point is at rest and the motion of every point of the circle for the moment is the same as if it described a circle about the point O. Hence the normal to the locus of P must pass through O, *i.e.*, PO is the normal, and its tangent must always be parallel to LP.

The same result also follows from the equations (1).

For, $\dfrac{\partial x}{\partial \theta} = a(1 - \cos\theta)$ and $\dfrac{\partial y}{\partial \theta} = a\sin\theta$

$$\therefore \quad \frac{\partial y}{\partial x} = \frac{\sin\theta}{1 - \cos\theta} = \cot\frac{\theta}{2} = \cot\tfrac{1}{2}\angle PCO = \cot PLO$$

The tangent, therefore, makes with the -axis an angle, the complement of PLO, and accordingly, LP is the tangent and PO is the normal to the curve at P.

Further, we have—

$$\left(\frac{\partial s}{\partial \theta}\right)^2 = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 = a^2\{(1 - \cos\theta)^2 + \sin^2\theta\}$$

$$= 4a^2\sin^2\frac{\theta}{2}$$

$$\therefore \quad \frac{\partial s}{\partial \theta} = 2a\sin\frac{\theta}{2} = PO = \text{length of the normal.}$$

261. The radius of curvature and evolute :

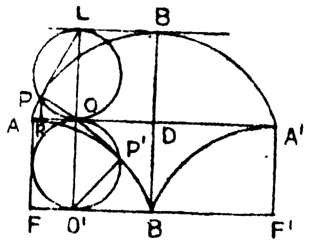
If ρ denotes the radius of curvature at the point P and

$$\angle \mathrm{POA}=\phi=\frac{\theta}{2}.$$

Then
$$\rho=\frac{\partial s}{\partial \phi}=2\frac{\partial s}{\partial \theta}=4a\sin\frac{\theta}{2}=2\mathrm{PO}$$

i.e., the radius of curvature is double the normal.

In order to determine the centre of curvature, we produce PO to P′, such that OP′=OP. Then P′ is the centre of curvature at P. Produce LO to O′, until OO′=OL. Describe a circle through O, P′ and O′. This circle touches AA′ and is equal to the generating circle LPO.



Now, the arc OP′=arc OP=AO.

$\therefore$ arc O′P′=O′P′O−P′O=AD−AO=OD=B′O′

Hence the locus of P′, *i.e.*, the evolute of the cycloid is a curve generated by the rolling of this new circle along the line B′O′, which is, therefore, a cycloid.

The same result also follows analytically.

If (x', y') be the co-ordinates of P′, *i.e.*, the centre of curvature, we have—

$$x'=a(\theta+\sin\theta), \qquad y'=a(-1+\cos\theta)$$

whence, by eliminating θ, we obtain the equation of the evolute.

But θ is an independent variable and we may easily replace θ by $\pi+\theta$, since this does not alter the nature of the curve. The equations thus reduce to—

$$x'=a(\theta+\pi-\sin\theta),\ y'=a(-1-\cos\theta)$$

If now we transfer the origin to the point B′, whose co-ordinates are $(\pi a, -2a)$, we obtain—

$$x'=a(\theta-\sin\theta), \qquad y'=a(1-\cos\theta)$$

whose locus is evidently an equal cycloid.

Hence *the evolute of a cycloid is another equal cycloid, having B′ as a cusp and A, A′ as vertices.*

It is easily seen that the evolute of the cycloid AB′A consists of the two semi-cycloids AB′ and B′A, as is shown in the figure. Conversely, the cycloid ABA′ is an involute of the cycloid AB′A′.

262. THEOREM:

The arc BP of the cycloid is double the chord LP of the circle, and the area of the cycloid is three times the area of the generating circle.

If α and β be the limits of θ, the arc s is obtained, as usual, by the formula—

$$s=\int_\alpha^\beta\sqrt{dx^2+dy^2}=2a\int_\alpha^\beta\sin\frac{\theta}{2}\,d\theta$$

$$=4a\left(\cos\frac{\alpha}{2}-\cos\frac{\beta}{2}\right)$$

If now we take $\beta=\pi$, we obtain the length of the arc

$$PB=4a\cos\frac{a}{2}.$$

But $$PL=2a\cos\frac{a}{2}$$

$$\therefore\quad s=\text{arc } PB=2\times\text{chord } PL.*$$

If we take $a=0$ and $\beta=2\pi$, $s=8a$, *i.e.*, the length of the complete arc ABA′ of the cycloid is equal to eight times the radius of the generating circle.

Again, the area of the portion PMA is given by—

$$S=a^2\int_0^\theta(1-\cos\theta)^2d\theta=a^2[\tfrac{3}{2}\theta-2\sin\theta+\tfrac{1}{4}\sin 2\theta]$$

If now $\theta=2\pi$, we obtain the area between the base and the complete arc ABA′

$$=a^2[\tfrac{3}{2}]2\pi=3\pi a^2$$

i.e., the area of the curve is three times that of the generating circle.

Ex. 1. If a circle roll on a right line, the envelope of any diameter is a cycloid.

Ex. 2. Show that the radius of the generating circle of the cycloid in Ex. 1 is half of the rolling circle.

Ex. 3. Prove that the envelope of any right line, carried by a circle which rolls on a right line, is the involute of a cycloid

Ex. 4. Show that the intrinsic equation of the cycloid is $s=4a\sin\phi$.

Ex. 5. Prove that $s=\sqrt{8ay}$, where y is the distance of P from NQ.

Ex. 6. If two tangents to a cycloid intersect at a constant angle, prove that the length of the portion which they intercept on the tangent at the vertex of the cycloid is constant.

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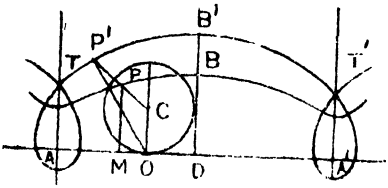
* Huygens—*Oeuvres*, Vol. 432, p. 3.

263. TROCHOIDS:

From the mode of generation of the cycloids it is natural to investigate the locus of a point in the plane of the generating circle carried round with it. Such loci are usually called *Trochoids.*

When the tracing point lies inside the circle, the locus is called the *prolate* cycloid, when it lies outside the circle, it is called the *curtate* cycloid. The forms are exhibited in the figure.



Let (x, y) be the co-ordinates of the tracing point P, referred to the axes AD and AT, where A is the position for which the moving radius CP is perpendicular to the fixed line.

Let a be the radius of the circle, b be the radial distance of P from the centre C, and $\angle \text{OCP}=\theta$, then we have

$$\left.\begin{aligned} x&=\text{AM}=\text{AO}-\text{OM}=a\theta-b\sin\theta \\ y&=\text{PM}=a-b\cos\theta \end{aligned}\right\} \quad \cdots \quad (1)$$

Eliminating θ between these, we obtain the Cartesian equation of the curve, which is a curtate or a prolate cycloid according as b is greater or less than a.

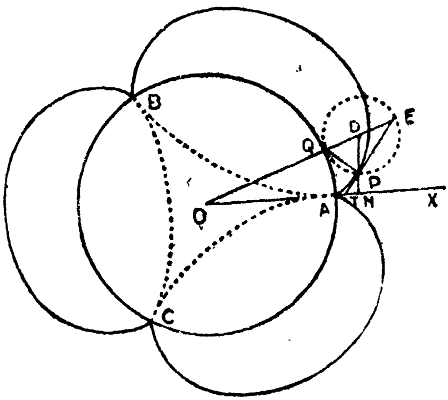
The method of drawing tangents given for the cycloid applies equally to the trochoids. It can be easily seen that these curves may be generated by a point P on the circumference of a circle, rolling such that the arc PO is in a constant ratio to the line AO.

264. EPICYCLOID AND HYPOCYCLOID:

As a natural extension, it is necessary to discuss the curves traced by a point connected with a circle rolling on the circumference of another circle instead of a line.

When the point is on the circumference of the rolling circle, the curve generated is called an *Epicycloid* or *Hypocycloid*, according as the rolling circle touches the exterior or interior of the circumference of the fixed circle. If, however, the generating point be not on the circumference, the locus is called an *Epitrochoid* or *Hypotrochoid.*

Let the radii of the rolling circle and of the fixed circle be respectively a and b. Let O be the centre of the fixed circle, Q the point of contact of the rolling circle and P the carried point. Suppose A is the point with which P is originally in contact, D the centre of the rolling circle. Join OD, cutting the rolling circle in E. Join QP, DP and EP, the latter cutting the initial radius OA in T. Take



OA as the axis of x. The point P traces the epicycloid as represented in the figure, which represents the curve when $a:b=1:3$.

Let $\angle QOA=\theta$, $\angle QDP=\phi$ and $\angle PTX=\psi$.

Now, since arc QP=arc QA, we have $b\phi=a\theta$

$$\therefore \quad \angle DEP=\frac{\phi}{2}=\frac{a\theta}{2b}$$

and
$$\psi=\theta+\frac{\phi}{2}=\frac{a+2b}{2b}\theta.$$

Again, DP makes with the x-axis the angle

$$\theta+\phi=\frac{a+b}{b}\theta.$$

Hence the equations of the Epicycloid are—

$$\left.\begin{aligned} x&=ON=OL+NL=(a+b)\cos\theta-b\cos(\theta+\phi)\\ &=(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta\\ y&=PN=DL-DP=(a+b)\sin\theta-b\sin(\theta+\phi)\\ &=(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta \end{aligned}\right\} \quad \ldots \quad (1)$$

where L is the foot of the ordinate of D.

In the case of the hypocycloid, the generating circle rolls on the interior of the fixed circle, and it can be easily seen that the expressions for x and y are derived from those in (1) by changing the sign of b. Thus

$$\left.\begin{aligned} x&=(a-b)\cos\theta+b\cos\frac{a-b}{b}\theta\\ y&=(a-b)\sin\theta-b\sin\frac{a-b}{b}\theta \end{aligned}\right\} \quad \ldots \quad (2)$$

When the radius of the rolling circle is a multiple of that of the fixed circle, the tracing point, after the circle has

rolled once round the circumference of the fixed circle, evidently returns to the same position and will trace the same curve in the next revolution. If the ratio of the radii is commensurable, the tracing point, after a certain number of revolutions, will return to its original position ; in this case, the curve is closed and of finite algebraic dimensions. On the other hand, if the ratio is incommensurable, the point will never return to the same position, but will describe an infinite series of distinct arcs, and the curve is transcendental. The successive portions of the curve are, however, equal to each other in every respect ; the path described by the tracing point, from the position in which it leaves the fixed circle until it returns to it again, is often taken instead of the complete epicycloid, and the middle point of this is called the *vertex* of the curve.

It is to be noted, however, that the points A, B, C are cusps on the epicycloid ; in fact, every point in which the tracing point meets the fined circle is a cusp on the roulette. It follows therefore that if the ratio $b : a = n$, there are n cusps on the corresponding epicycloid or hypocycloid; and thus the se curves are designated by the number of their cusps, such as the three-cusped, four-cusped, etc., epi-, or hypo-cycloids.

265. Double generation of Epi - and Hypo-Cycloids:

If in the equations (2) of the hypocycloid, we write $\frac{a+d}{2}$ for b, and $\frac{a+d}{d}\phi$ for θ, we obtain—

$$\left.\begin{aligned} x&=\frac{a-d}{2}\cos\frac{a+d}{d}\phi+\frac{a+d}{2}\cos\frac{a-d}{d}\phi \\ y&=\frac{a-d}{2}\sin\frac{a+d}{d}\phi-\frac{a+d}{2}\sin\frac{a-d}{d}\phi \end{aligned}\right\} \quad \ldots \quad (3)$$

and it is evident that a change in the sign of d does not alter these equations. Hence it follows that the same

hypocycloid can be generated by the rolling of either of the circles whose radii are $\frac{a \pm d}{2}$ on a circle of radius a.

Again, if we write $a+d$ for b and $\frac{a+d}{d}\phi$ for θ, in the equations (2), the equations of the hypo-cycloid reduce to—

$$\left.\begin{aligned} x &= (a+d)\cos\phi - d\cos\frac{a+d}{d}\phi \\ y &= (a+d)\sin\phi - d\sin\frac{a+d}{d}\phi \end{aligned}\right\} \quad \ldots \quad (4)$$

which are evidently the equations of an epicycloid. It appears, therefore, that when the radius of the rolling circle is greater than that of the fixed circle, the hypocycloid generated may be regarded as an epicycloid generated by the rolling of a circle whose radius is the difference of the original radii.* The same results may be obtained from geometrical considerations, for which the student is referred to Williamson's Diff. Calc., Art. 280.

By writing

$$m = (a+d)/d$$

these equations can be written as—

$$x = d(m\cos\phi \pm \cos m\phi)$$

$$y = d(m\sin\phi \pm \sin m\phi)$$

the lower sign answering to the case when the axis of x passes through the generating point when it is on the fixed circle; the upper sign, when it is at its greatest distance from it.

* The student should carefully note the anomaly that every epicycloid is a hypocycloid but only some hypocycloids are epicycloids; while, in fact, no epicycloid is a hypocycloid, though each can be generated in two ways as has been shown. For correct representation, see *Proctor's Geometry of Cycloids.*

266. PARTICULAR CASES:

As has already been remarked, if the ratio $a : b$ is commensurable, there will be a finite number of cusps, the curve returning into itself.

If, for example, $a=2b$, the equations (2) of the hypocycloid become—

$$x=a\cos\theta, \qquad y=0,$$

which represents a diameter of the fixed circle.

Thus, in this case the hypocycloid reduces to a diameter of the fixed circle.

If $a=4b$, we have—

$$\left.\begin{aligned} x&=\tfrac{a}{4}(3\cos\theta+\cos 3\theta)=a\cos^3\theta \\ y&=\tfrac{a}{4}(3\sin\theta-\sin^3\theta)=a\sin^3\theta \end{aligned}\right\}$$

whence eliminating θ, we obtain the equation—

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$

which represents a four-cusped hypocycloid.

If in the equations (1) of the epicycloid, we put $b=a$, the equations become—

$$x=a(2\cos\theta-\cos 2\theta)$$

$$y=a(2\sin\theta-\sin 2\theta)$$

whence, eliminating θ, we obtain—

$$(x^2+y^2-a^2)^2=4a^2\{(x-a)^2+y^2\}$$

and when the origin is transferred to the point $(a, 0)$, the equation becomes—

$$(x^2+y^2+2ax)^2=4a^2(x^2+y^2)$$

which is a Cardioid.

Ex. 1. Obtain the equation of the epicycloid when the radius of the rolling circle is half that of the fixed circle.

Ex. 2. When the incident rays are parallel, show that the caustic by reflexion at a circle is an epicycloid formed by the rolling of one circle upon another of twice its radius.

Ex. 3. A is a fixed point on the circumference of a circle. Points L and M are taken such that arc AL$=m$ arc AM, where m is a constant. Prove that the envelope of LM is an epicycloid or hypocycloid, according as the arcs AL and AM are measured in the same or opposite directions from the point A.

Ex. 4. Show that in *Ex.* 3 LM is divided at the point of contact in the ratio $m : 1$ internally or externally according as the envelope is an epicycloid or hypocycloid.

Ex. 5. Show that in *Ex.* 3 the given circle is circumscribed to or inscribed in, the envelope, according as it is an epicycloid or hypocycloid.

267. Radius of curvature :

From the equations (1) of the curve, we obtain—

$$\frac{\partial^2 x}{\partial\theta^2}=(a+b)\left\{-\cos\theta+\left(1+\frac{a}{b}\right)\cos(\theta+\theta')\right\}$$

$$\frac{\partial^2 y}{\partial\theta^2}=(a+b)\left\{-\sin\theta+\left(1+\frac{a}{b}\right)\sin(\theta+\theta')\right\}$$

where $$\theta'=\frac{a}{b}\theta.$$

The radius of curvature ρ may be calculated from the formula—

$$\rho=\frac{\left\{\left(\frac{\partial x}{\partial\theta}\right)^2+\left(\frac{\partial y}{\partial\theta}\right)^2\right\}^{\frac{3}{2}}}{\frac{\partial x}{\partial\theta}\cdot\frac{\partial^2 y}{\partial\theta^2}-\frac{\partial y}{\partial\theta}\cdot\frac{\partial^2 x}{\partial\theta^2}}=\frac{4(a+b)a\sin\frac{1}{2}\theta'}{a+2b}$$

$$\therefore\quad \rho=\frac{2k(a+b)}{a+2b},\qquad \text{where}\quad k=\text{QP}.$$

Thus the radius of curvature in an epicycloid is in a constant ratio to the chord QP joining the generating point to the point of contact of the circles.

Since the denominator in ρ can never be zero, it follows that the curve has no points of inflexion.

268. THE EVOLUTE OF AN EPICYCLOID:

We shall now show that the evolute of an epicycloid is a similar epicycloid.*

Let P be the tracing point in any position, and A its position on the fixed circle.

Join P to O the point of contact of the circles, and produce PO to P′, such that

$$PP'=OP\,\frac{2a+2b}{a+2b}.$$

Consequently, P′ is the centre of curvature, and

$$\therefore \quad OP'=OP.\,\frac{a}{a+2b}$$

Now, draw P′O′ perpendicular to P′O and circumscribe a circle about the triangle OP′O′, and describe a circle with C as centre and CO′ as radius. This circle evidently touches the circle OP′O′.

But $\quad OO' : OE=OP' : OP=a : a+2b=CO : CE$

$$\therefore \quad CO-OO' : CE-OE=CO : CE$$

or $\quad CO' : CO=CO : CE$

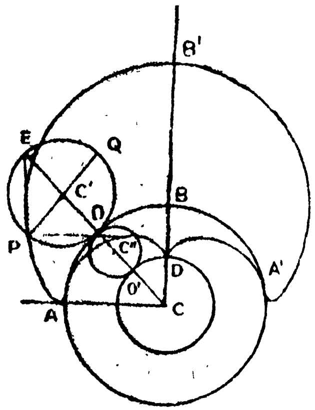
i.e., the lines CE, CO and CO′ are in geometrical proportion.

Now, join C to B′, the vertex of the epicycloid and let CB′ meet the inner circle at D.

* This proof is taken from Williamson's Diff. Calc., § 281.

Then, arc O′D : arc OB $=$ CO′ : CO $=$ CO : CE

$=$ O′O : EO $=$ arc P′O′ : arc OQ.



But arc OB $=$ arc OQ $\therefore$ arc O′D $=$ arc P′O′.

Hence, the path described by P′ is that generated by a point on the circumference of the circle OP′O′ rolling on the inner circle, and starting when P′ is in contact at D. Thus *the evolute of the original epicycloid is another epicycloid,* as is shown in the figure.

Since CO : OE $=$ CO′ : O′O, the ratio of the radii of the fixed and generating circles is the same for both the epicycloids, and consequently *the evolute is a similar epicycloid.*

The same result may be deduced analytically by writing the equations of the tangent and normal at any point.

For, $$\frac{\partial y}{\partial x}=\frac{\cos\phi\pm\cos m\phi}{-(\sin\phi\pm\sin m\phi)}=-\frac{\cos\frac{1}{2}(m+1)\phi}{\sin\frac{1}{2}(m+1)\phi}$$

$$\text{or, }\frac{\sin\frac{1}{2}(m+1)\phi}{\cos\frac{1}{2}(m+1)\phi}\cdot \quad \ldots \quad (1)$$

whence the equation of the tangent becomes—

$$x\cos\tfrac{1}{2}(m+1)\phi+y\sin\tfrac{1}{2}(m+1)\phi$$
$$=(m+1)b\cos\tfrac{1}{2}(m-1)\phi \quad \ldots \quad (2)$$

when the axis passes through the generating point at its greatest distance from the centre of the fixed circle;

or
$$x \sin \tfrac{1}{2}(m+1)\phi - y \cos \tfrac{1}{2}(m+1)\phi = (m+1)b \sin \tfrac{1}{2}(m-1)\phi \quad \ldots \quad (3)$$

when the x-axis passes through the generating point at its least distance from the centre of the fixed circle. In this latter case, the equation of the normal becomes—

$$x \cos \tfrac{1}{2}(m+1)\phi + y \sin \tfrac{1}{2}(m+1)\phi = (m-1)b \cos \tfrac{1}{2}(m-1)\phi \quad \ldots \quad (4)$$

Comparing this form with the equation (2), we see that the envelopes of (2) and (4) are of the same nature, in fact, they are similar epicycloids, since the radii of the circles are altered in the ratio $\dfrac{m-1}{m+1}$.

269. The reciprocal of an epicycloid: *

Since the equation of the tangent at any point may be written as—

$$x \cos \tfrac{1}{2}(m+1)\phi + y \sin \tfrac{1}{2}(m+1)\phi = (m+1)b \cos \tfrac{1}{2}(m-1)\phi$$

the perpendicular on the tangent makes an angle

$$\omega = \tfrac{1}{2}(m+1)\phi$$

with the x-axis and its length is—

$$(m+1)b \cos \tfrac{1}{2}(m-1)\phi.$$

* The invention of epicycloids is attributed to the Danish Astronomer, Roemer, who was led to consider these curves in the year 1674 for examining the best form of teeth of wheels. The rectification, etc., were given by Newton, *Principia*, Book I.

The locus of the foot of this perpendicular is then given by—

$$p=(m+1)b\cos\left(\frac{m-1}{m+1}\omega\right)$$

and the reciprocal curve is given by—

$$r\cos\left(\frac{m-1}{m+1}\omega\right)=(m+1)b$$

Ex. 1. Prove that the radius of curvature of an epicycloid varies as the perpendicular on the tangent from the centre of the fixed circle.

Ex. 2. Prove that the locus of intersection of tangents to a cycloid which intersect at a constant angle is a prolate trochoid.

Ex. 3. If a variable circle touch a given cycloid and also touch the tangent at the vertex, the locus of its centre is a cycloid. (Casey.)

Ex. 4. Prove that the equation of the reciprocal polar of an epicycloid with respect to the fixed circle is of the form $r\sin m\omega=\text{const.}$

270. Epitrochoid and Hypotrochoid :

The equations of these curves may be obtained as in § 264 in the following manner :

Let d be the constant distance of the generating point P from the centre O of the rolling circle of radius b.

Take for the axis of x that position of the common diameter of the two circles which passes through the generating point. Let C be the centre of the fixed circle of radius a, and D the point of contact in any position. Then we have $\qquad \text{CD}=a, \quad \text{OD}=b, \quad \text{OP}=d.$

Let OP meet the rolling circle at Q and the axis of x meet the fixed circle in A and B. From O and P draw OM and PN perpendiculars on ACB, and draw PE perpendicular on OM, and let

$$\angle\text{DCB}=\theta, \qquad \angle\text{QOD}=\phi.$$

Since $BD=DQ$, we must have $a\theta=b\phi$ and

$$OPE=180^\circ-(\theta+\phi)$$

$\therefore$ The co-ordinates of the point P are—

$$x=(a+b)\cos\theta-d\cos(\theta+\phi)$$

$$y=(a+b)\sin\theta-d\sin(\theta+\phi)$$

or

$$\left.\begin{aligned} x&=(a+b)\cos\theta-d\cos\frac{a+b}{b}\theta \\ y&=(a+b)\sin\theta-d\sin\frac{a+b}{b}\theta \end{aligned}\right\} \quad \ldots \quad (1)$$

If now we put $a+b=mb$, we obtain—

$$\left.\begin{aligned} x&=mb\cos\theta-d\cos m\theta \\ y&=mb\sin\theta-d\sin m\theta \end{aligned}\right\} \quad \ldots \quad (2)$$

Eliminating θ from these equations the equation of the curve can be obtained. These curves are not necessarily transcendental, and as has already been remarked, they are transcendental only when the circumferences of the circles are not commensurable.

The equations of the epicycloid can be obtained from (2) simply by putting $d=\pm b$. The equations of the hypotrochoid are obtained by changing the signs of b and d in (1).

Thus the equations of a hypotrochoid are obtained in the form :

$$\left.\begin{aligned} x&=(a-b)\cos\theta+d\cos\frac{a-b}{b}\theta \\ y&=(a-b)\sin\theta-d\sin\frac{a-b}{b}\theta \end{aligned}\right\} \quad \ldots \quad (3)$$

In the particular case when $a=2b$, *i.e.*, when a circle rolls inside another of double its radius, the equations (3) become—

$$x=(b+d)\cos\theta, \qquad y=(b-d)\sin\theta$$

whence the equation of the hypotrochoid is—

$$\frac{x^2}{(b+d)^2}+\frac{y^2}{(b-d)^2}=1$$

which represents an ellipse whose semi-axes are the sum and the difference of b and d. This reduces to the diameter y, when $b=d$ in the case of the hypocycloid.

Ex. 1. If the distance d of the generating point from the centre of the rolling circle be equal to the distance between the centres of the circles, prove that the polar equation of the epitrochoid becomes—

$$r=2(a+b)\cos\frac{a\theta}{a+2b}.$$

Ex. 2. Hence deduce that the curve $r=a\sin m\theta$ is an epitrochoid when $m<1$, and a hypotrochoid when $m>1$.

Ex. 3. Prove that the locus of the intersection of the tangents to an epicycloid which intersect at a constant angle is an epitrochoid and the corresponding locus for a hypocycloid is a hypotrochoid (Chasles).

271. THE THREE-CUSPED HYPOCYCLOID:

This curve, as already remarked, is traced out by a point on the circumference of a circle which rolls inside another circle of three times its radius.

If in the equations (2) of the hypocycloid (§ 264) we put $a=3b$, they reduce to—

$$\left.\begin{aligned} x/b&=2\cos\theta+\cos 2\theta \\ y/b&=2\sin\theta-\sin 2\theta \end{aligned}\right\} \quad \ldots \quad (1)$$

The Cartesian equation is obtained by eliminating θ between these equations.

Thus, squaring and adding, we get—

$$\tfrac{1}{4}(r^2-5b^2)=b^3\cos 3\theta=b^3\cos\theta(4\cos^2\theta-3) \quad \ldots \quad (2)$$

But
$$x+b=2b\cos\theta(1+\cos\theta) \quad \ldots \quad (3)$$

Dividing (2) by (3), we obtain—

$$\frac{r^2-5b^2}{2b(x+b)}=\frac{4\cos^2\theta-3}{1+\cos\theta} \quad \ldots \quad (4)$$

Substituting the value of $\cos^2\theta$ from (3) in (4), we get—

$$(1+\cos\theta)(r^2+8bx+3b^2)=2(x+b)(2x+3b) \quad \ldots \quad (5)$$

Now, eliminating θ between (3) and (5), we obtain—

$$(r^2+12bx+9b^2)^2=4b(2x+3b)^3$$

i.e.,
$$r^4+18b^2r^2-8bx^3+24bxy^2=27b^4$$

or
$$(x^2+y^2)^2+8bx(3y^2-x^2)$$
$$+18b^2(x^2+y^2)-27b^4=0 \quad \ldots \quad (6)$$

which is a tricuspidal quartic.

By putting $\tan\frac{1}{2}\theta=-t$, the equations (1) may be put into the following form—

$$x=b\,\frac{-t^4-6t^2+3}{(t^2+1)^2}, \qquad y=-b\frac{8t^3}{(t^2+1)^2}, \quad \ldots \quad (7)$$

whence
$$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial t}\cdot\frac{\partial x}{\partial t}=t.$$

The tangential equation of the curve is obtained in the form—

$$\xi^2+\eta^2=b\xi(3\eta^2-\xi^2)$$

showing that the line at infinity is the only bitangent touching the curve at the circular points. For further details, see G. Loria, *loc. cit.*, pp. 161-167, and Teixeira, pp. 442-457.

272. THE TANGENT TO A THREE-CUSPED HYPOCYCLOID:

Using the equations (1) of § 271, the equation of the tangent to the curve at any point (a) can be obtained in the form—

$$\{y-b(2\sin a-\sin 2a)\}$$

$$=\frac{\cos a-\cos 2a}{-\sin a-\sin 2a}\{x-b(2\cos a+\cos 2a)\}$$

which easily reduces to—

$$x\sin\frac{a}{2}+y\cos\frac{a}{2}=b\sin\frac{3a}{2} \qquad \ldots \quad (1)$$

Transferring the origin to the cusp A, we put $x-3b$ for x and obtain from (7) of § 271

$$x=-\frac{4bt^2(t^2+3)}{(t^2+1)^2}, \qquad y=-b\frac{8t^3}{(t^2+1)^2}.$$

The equation of the tangent at any point (x, y) is—

$$Y-y=t(X-x)$$

which referred to the new origin becomes—

$$Y-tX=y-tx=\frac{4bt^3}{t^2+1}$$

If this passes through any point (x', y') we have—

$$(4b+x')t^3-y't^2+x't-y'=0 \qquad \ldots \quad (2)$$

This is a cubic equation in t and gives the slopes of the tangents, all three real, or one real and two imaginary, which can be drawn from (x', y') to the curve.

If now we put $t_1=\tan a$, $t_2=\tan\beta$, $t_3=\tan\gamma$

we have $\quad t_1+t_2+t_3=\dfrac{y'}{4b+x'} \quad$ and $\quad t_1t_2t_3=\dfrac{y'}{4b+x'}$.

whence we obtain—

$$\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha . \tan \beta . \tan \gamma .$$

i.e.,
$$\tan \gamma = -\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = -\tan (\alpha + \beta).$$

or
$$\alpha + \beta + \gamma = n\pi .$$

If now we take the cuspidal tangent as the axis of x, we obtain the theorem.*

The three tangents drawn from any external point to a three-cusped hypocycloid make with a cuspidal tangent three angles whose sum is a multiple of π.

The condition that two of these tangents are at right angles is that $t_1 t_2 + 1 = 0$.

But since $t_1 t_2 t_3 = \dfrac{y'}{4b + x'}$, we obtain—

$$t_3 = -\frac{y'}{4b + x'} .$$

Again, t_3 is a root of the equation (2). Substituting this value of t in that equation, we obtain—

$$\frac{2y'^2}{(4b + x')^2} + \frac{x'}{4b + x'} + 1 = 0 \qquad \dots \quad (3)$$

If now we consider (x', y') as variables, this represents the locus of points from which perpendicular tangents can be drawn to the hypocycloid. The equation can, however, be written in the form—

$$(x' + 3b)^2 + y'^2 = b^2$$

We thus obtain the theorem of Painvin:

The locus of points from which perpendicular tangents can be drawn to a three-cusped hypocycloid is a circle of radius b and centre coinciding with that of the hypocycloid.

* Laguerre—*Nouvelles Annales de Mathématiques* (1870), p. 254.

273. THE CONVERSE THEOREM : *

Every quartic curve of class three having the line at infinity as the bitangent with the points of contact at the circular points is a three-cusped hypocycloid.

Consider the equation $\xi\,\eta\,\zeta = \xi^3+\eta^3$ ... (1)

where ξ, η, ζ are the co-ordinates of a line. This represents a curve of class three, having the third side of the triangle of reference as a bitangent, with the corresponding vertices as points of contact.

This equation may be expressed in the following parametric form—

$$k\xi=\lambda^2,\quad k\eta=\lambda,\quad k\zeta=1+\lambda^3$$

$\therefore$ The equation of a tangent to the curve may be written as—

$$\lambda^2 x+\lambda y+(1+\lambda^3)z=0 \qquad \text{... (2)}$$

Differentiating this *w.r.t.* λ, we get—

$$2\lambda x+y+3\lambda^2 z=0 \qquad \text{... (3)}$$

Combining (2) and (3), the co-ordinates of any point on the curve are found to be proportional to $1-2\lambda^3 : \lambda^4-2\lambda : \lambda^2$

If now we put $\lambda=(\mu/\nu)$ then—

$$\frac{x}{\nu^4-2\mu^3\nu}=\frac{y}{\mu^4-2\nu^3\mu}=\frac{z}{\mu^2\nu^2}$$

If again $\mu=\rho e^{\frac{i\theta}{2}}$, $\nu=\rho e^{-\frac{i\theta}{2}}$, $\dfrac{x+y}{2z}=\dfrac{X}{a}$, $\dfrac{y-x}{2iz}=\dfrac{Y}{a}$.

Then $X=a(\cos 2\theta+2\cos\theta)$, $Y=a(\sin 2\theta-2\sin\theta)$.

This, by suitable transformation, may be put in the forms :

$$X=a(\cos 2\theta+2\cos\theta),\qquad Y=a(2\sin\theta-\sin 2\theta)$$

which are the equations of a three-cusped hypocycloid.

* L. Cremona—*Sur l'hypocycloide à trois rebroussements*—Crelle, Bd. 64 (1865).

274. THEOREM: *

The envelope of the pedal line of a triangle is a three-cusped hypocycloid, the centre of which is the centre of the nine-points circle.

Let P be any point on the circumscribing circle with centre O, and let PK, PL be the perpendiculars on the sides AC, AB and NY the perpendicular from N, the middle point of AC, on KL, which is called the *pedal* line.

Let $\angle ONY=\phi$ and $\angle COP=\theta$

Then, $(\pi/2)-\phi=\angle YNK=\angle LKP=\angle LAP=A-\frac{1}{2}\theta$

and $NK=OP\sin OPK=R\sin(\pi-\theta-B)$

$$=R\sin(\theta+B)=R\sin(B+2A+2\phi-\pi)$$

$\therefore$ $NY=NK\sin\phi$

$$=R\sin\phi\sin(B+2A+2\phi-\pi),\quad \text{where } OP=R$$

$$=(R/2)[\cos\{\phi-(C-A)\}-\cos\{3\phi-(C-A)\}]$$

Now, if O′ be the centre of the nine-points circle,

$$\angle ONO'=C-A$$

and if p be the perpendicular from O′ on LK,

$$p=-\frac{R}{2}\cos\{3\phi-(C-A)\}=-\frac{R}{2}\cos 3\phi',$$

changing the initial line.*

This is the tangential polar equation of a three-cusped hypocycloid, generated by a circle of radius $(R/2)$, rolling inside a circle of radius$(3R/2)$.†

* Besant—*Notes on Roulettes and Glisettes* § 26(3). This theorem seems to have been first discovered by Steiner.

† Edward's Diff. Calc., § 410.

275. THEOREM:

If a rectangular hyperbola circumscribe a triangle, the envelope of its asymptotes is a three-cusped hypocycloid.

Let O_1 be the centre of the hyperbola, O' that of the nine-points circle of the triangle ABC, and N and M the middle points of the sides AC and AB. Then from a known property, the point O_1 lies on the nine-points circle; and if D be taken on AC such that $ND=O_1N$, O_1D is an asymptote of the hyperbola.*

Draw $O'Q$ perpendicular to O_1D and let $O'Q=p$ and $\angle O_1DN=\phi$.

Then, $$p=\tfrac{1}{2}R\cos O_1O'Q$$

and $$O_1O'Q=\tfrac{1}{2}\pi-O'O_1N+\phi=O_1MN+\phi$$

Again, since $\angle MO_1N=A$,

$$\angle O_1MN=\pi-A-O_1NM$$

$$=C-A+2\phi$$

$$\therefore\quad O_1O'Q=C-A+3\phi$$

Hence, $$p=\tfrac{1}{2}R\cos(3\phi+C-A)$$

which represents a three-cusped hypocycloid.

Again, the centre of a rectangular hyperbola with respect to which a triangle is self-conjugate lies on the circum-circle of the triangle, and the curve passes through the centres of the inscribed and the escribed circles of the triangle. Hence we obtain the theorem:

The envelope of the asymptotes of all rectangular hyperbolas to which a given triangle is self-conjugate is a three-cusped hypocycloid, whose centre is at the circum-centre of the triangle.

For other properties of a three-cusped hypocycloid, see G. Loria, *loc. cit.*, pp. 161-167, and Teixeira, pp. 442-457.

* Salmon—*Conics*, § 174.

Ex. 1. Show that the tangent to the three-cusped hypocycloid at any point (α) meets the curve again in the two points

$$\left(\pi-\frac{\alpha}{2}\right) \quad \text{and} \quad \left(2\pi-\frac{\alpha}{2}\right).$$

Ex. 2. Show that the sum of the parameters of the points of contact of three concurrent tangents to a three-cusped hypocycloid is equal to four right angles.

Ex. 3. Obtain the equation of the normal at any point (α) of the three-cusped hypocycloid in the form—

$$x \cos\frac{\alpha}{2}-y \sin\frac{\alpha}{2}=3b \cos\frac{3\alpha}{2}.$$

Ex. 4. Shew that the evolute of a three-cusped hypocycloid is a similar three-cusped hypocycloid, and the orthoptic locus is a circle.

Ex. 5. The radius of curvature at any point (α) of a three-cusped hypocycloid is $8b \sin(3\alpha/2)$.

Ex. 6. Show that the portion of the tangent of the three-cusped hypocycloid intercepted by the curve is of constant length.

276. The four-cusped hypocycloid :

If in the equations (2) of § 264 we put $a=4b$,

the equations of the curve become—

$$x=3b \cos\theta+b \cos 3\theta=a \cos^3\theta$$

$$y=3b \sin\theta-b \sin 3\theta=a \sin^3\theta$$

Eliminating θ between these, the equation of the four-cusped hypocycloid is obtained in the form—

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}} \qquad \ldots \quad (1)$$

or, rationalising this becomes—

$$(a^2-x^2-y^2)^3=27a^2x^2y^2 \qquad \ldots \quad (2)$$

The characteristics of the curve can be easily investigated by means of the reciprocal curve, whose equation can be written in the form :

$$x^{-2}+y^{-2}=c^{-2}.$$

The curve has a complex biflecnode at the origin and a pair of real biflecnodes at infinity situated on the axes of x and y respectively. It is of order six and class four, and consequently belongs to the seventh species. The characteristics of the curve are therefore $n=6$, $\delta=4$, $\kappa=0$, $m=4$, $\tau=0$, $\iota=0$.

All the four nodes are imaginary and situated on the lines $x\pm y=0$.

277. THEOREM :

The envelope of a straight line of constant length, which slides between two straight lines at right angles to one another, is a four-cusped hypocycloid.

Let a straight line AB of constant length a slide between two lines at right angles intersecting at O. Let C be the mid-point of AB, and join OC and produce OC to D making OC=CD. From O and D draw OL and DP perpendiculars to AB. Now, since DA, DB are evidently perpendicular to OA and OB, PB is the direction of motion of P, and consequently AB touches its envelope at P.

Let (x, y) be the co-ordinates of P, and $\angle$ AOL$=\theta$.

Then, $\quad x=\text{BP}\sin\theta=\text{AL}\sin\theta$

$$=a\sin^3\theta \quad \text{and} \quad y=a\cos^3\theta$$

$\therefore$ The locus of P is obtained in the form—

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

The lines OA and OB are the two real cuspidal tangents, and we may accordingly state the theorem :

The portion of the tangent to a four-cusped hypocycloid which is intercepted by two real cuspidal tangents is of constant length.

278. The Catenary:

The Catenary is the form assumed by a flexible inelastic string of uniform density, when suspended from two points under the action of gravity.

The form of equilibrium of a flexible chain was first investigated by Galileo, who declared that the curve was a parabola. His error was detected in 1669 experimentally by Joachim Jungius, a German geometer. The true form of the catenary was, however, obtained by James Bernoulli only in 1691.

The equation of the curve may be easily deduced from mechanical considerations.

Let A, B be the points of suspension and O the lowest point of the string. Let T be the tension at any point P of the curve. If then w is the weight of a unit of length, wc is the tension at the lowest point O, where c is a constant.

Let s be the length of the arc OP and ψ the angle which the tangent at P makes with the horizontal line through O.

Then the equations of equilibrium of OP are

$$T \cos \psi = wc, \qquad T \sin \psi = ws,$$

$$\therefore \quad s = c \tan \psi \qquad \cdots \quad (1)$$

which is the *intrinsic* equation of the curve.

Take the vertical line through O as the axis of y and the horizontal line at a distance c from the lowest point O as the axis of x.

$$\text{Now,} \quad \tan \psi = \frac{\partial y}{\partial x} \quad \text{and} \quad \partial s^2 = \partial x^2 + \partial y^2$$

$$\therefore \quad s^2 + c^2 = c^2 \left(\frac{\partial s}{\partial x} \right)^2 \quad \text{or} \quad \partial x = \frac{c \partial s}{\sqrt{s^2 + c^2}}$$

$$\therefore \quad \frac{x}{c} = \log \left\{ \frac{x + \sqrt{(s^2 + c^2)}}{c} \right\} \qquad \ldots \quad (2)$$

the constant being taken so that s and x vanish together.

From (2) we may write the equation of the catenary in the form :

$$e^{\frac{x}{c}} + e^{-\frac{x}{c}} = \frac{2\sqrt{(s^2+c^2)}}{c}$$

and $$e^{\frac{x}{c}} - e^{-\frac{x}{c}} = \frac{2s}{c} \quad \dots \quad (3)$$

Similarly, we obtain from the equation of the curve

$$\frac{s^2+c^2}{s^2} = \left(\frac{\partial s}{\partial y}\right)^2 \qquad i.e., \ \partial y = \frac{s\partial s}{\sqrt{(s^2+c^2)}}$$

$\therefore \quad y^2 = s^2 + c^2$, if $y=c$ when s or $x=0$.

Thus we get

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) = c\cosh\frac{x}{c} \quad \dots \quad (4)$$

which is the Cartesian equation of the curve.

Also $$s = c\sinh\frac{x}{c}.$$

The equation (4) shows that the axis of y is the axis of symmetry of the curve and the tangent at the lowest point is parallel to the axis of x. As x varies from 0 to $\pm\infty$, y varies from c to ∞, and the curve is convex to the axis of x.

279. Construction of the tangent to the catenary :

From the equation

$$\sqrt{y^2-c^2} = s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)$$

we obtain,

$$\frac{\partial y}{\partial x} = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) = \frac{\sqrt{(y^2-c^2)}}{c}$$

With the origin C as centre and radius c describe a circle and draw the tangent MT to this circle from the point

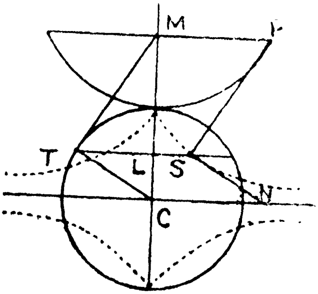
M on the axis of y at a distance from the axis of x equal to the ordinate of the point P. Then

$$MC=y, \quad CT=c, \quad MT=\sqrt{(y^2-c^2)}$$

If L is the point where TS drawn parallel to the x-axis meets MC, we have

$$\tan \angle MCT=\tan \angle MTS$$

$$=\frac{\sqrt{(y^2-c^2)}}{c}=\frac{\partial y}{\partial x}$$



Thus, the tangent PS at P is parallel to MT.

Also PS=MT=the arc from P to the lowest point. The axis of x is called the *directrix* of the catenary. The locus of the point S is the involute of the catenary; and if SN is drawn parallel to TC, SN is the tangent to the involute, since PS is normal to the locus of S The involute of the catenary is consequently a curve such that the intercept SN on its tangent between the point of contact and a fixed right line is constant, and is called a *Tractrix*.

Let the tangent and the normal at P meet the axis of x in G and H respectively. PN is evidently the ordinate of P and NS is perpendicular to PG. Then

$$\frac{\partial y}{\partial x}=\tan\psi=\frac{1}{c}\frac{\partial y}{\partial\psi}\cos\psi$$

whence $\quad y=c\sec\psi \qquad$ Also $\quad y=PN=NS\sec\psi$

$\therefore$ NS$=c$ and $s=c\tan\psi=$NS $\tan\psi=$PS, showing that the locus of S is the involute of the catenary.

If ρ be the radius of curvature at P, then

$$\rho=(\partial s/\partial\psi)=c\sec^2\psi=\text{PN}\sec\psi=\text{PH}$$

i.e., the centre of curvature at P is a point O′ on HP produced such that PO′$=$PH.

Thus the radius of curvature at any point is equal but opposite to the normal at that point.

280. THEOREM :

If a parabola roll on a straight line, the locus of its focus is a catenary.

Let F be the focus of the parabola in one of its positions, touching the fixed line OP at P. Since the curve rolls without sliding, the point P may be supposed to remain stationary for the time being, so that FP rotates about P, and the direction of motion of F at that instant is at right angles to FP. Thus the line FO, drawn at right angles to FP, is the tangent to the locus of F. Let FO make an angle ψ with the fixed line, which is taken as the x-axis.

Draw the ordinate FM, and let (x, y) be the co-ordinates of F.

Then, $$\cos\psi=(\partial x/\partial s)=(\text{FM}/\text{FP})$$

But from the equation of the parabola, we have—

$$p^2=ar, \qquad \textit{i.e.,} \quad \text{FM}^2=a.\text{FP}$$

$$\therefore \quad (\partial x/\partial s)=(a.\text{FM}/\text{FM}^2)=(a/\text{FM})=(a/y)$$

whence $$a(\partial y/\partial x)=\sqrt{y^2-a^2}$$

and $$(x/a)=\log_e(y+\sqrt{y^2-a^2})+\text{C}$$

Let $x=0$, when $y=a$ $\quad\therefore\quad$ $\text{C}=-\log_e a$

Thus $$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)=a\cosh\frac{x}{a}$$

which is a catenary.

281. THE TRACTRIX:

We have already seen that the involute of a catenary is a curve called the Tractrix or Tractory. The equation of this curve may be obtained without much difficulty.

Let (x, y) be the co-ordinates of S. If Q be the of the ordinate of S, we have

$$QN^2 = SN^2 - SQ^2 \qquad \therefore \quad QN = \sqrt{c^2 - y^2}$$

By making $y=0$ in the equation of the tangent, we find that this length $= -(y\partial x/\partial y)$.

Hence the differential equation becomes—

$$-\frac{y\partial x}{\partial y} = \sqrt{(c^2 - y^2)}$$

This may be easily made rational by putting $z^2 = c^2 - y^2$, and we obtain

$$\partial x = \frac{c^2 \partial z}{c^2 - z^2} - \partial z$$

$$\therefore \quad x = c \log \left\{ \frac{c+z}{(c^2 - z^2)} \right\} - z$$

i.e.

$$x = c \log \left\{ \frac{c + \sqrt{(c^2 - y^2)}}{y} \right\} - \sqrt{(c^2 - y^2)}$$

This curve is easily found to consist of four similar portions, as shown by the dotted curve in the figure of § 279.

The tractrix is a particular case of the general problem of equitangential curves, *i.e.*, curves such that the intercept on the tangent between the curve and a fixed directrix shall be constant.

In order to obtain the intrinsic equation, we see that $y = c\cos\psi$ and $SP = c\tan\psi = (\partial s/\partial\psi)$; whence $s = c\log\sec\psi$, which is the intrinsic equation of the curve.

The locus of a point D on the tangent SN dividing it in a constant ratio is called the *Syntractrix*.

282. THE INVOLUTE OF A CIRCLE.

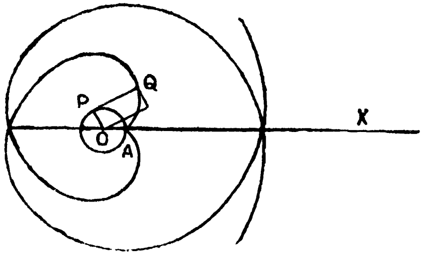
The locus traced by a point in the plane of a straight line when the line rolls on the circle is called an *involute* of the circle. The locus may be represented by the equations *

$$x=(a+t)\cos\phi+a\phi\sin\phi$$

$$y=(a+t)\sin\phi-a\phi\cos\phi$$

where t is the length of the perpendicular drawn from the point on the line.

For $t=0$, we obtain the ordinary cusped involute, which is a transcendental curve. If a thread be unwound from a circle, any point of the unwinding thread traces out an involute of the circle whose equation can be easily obtained.



If PQ be the tangent at any point P of a circle of radius a, and on this tangent a portion PQ be taken such that it is equal to the arc AP measured from a fixed point A, then the locus of Q is the involute of the circle.

Let O be the centre of the circle and $OQ=r$, $\angle POA=\phi$, $\angle QOA=\theta$.

Then, $OQ\cos QOP=a$, *i.e.*, $r\cos(\phi-\theta)=a$ ... (1)

* The locus may be obtained as a particular case of an epitrochoid by considering the rolling circle as of infinite radius, *i.e.*, putting $d=b=\infty$ and $t=b-d$ in the equations (1) of § 270.

Also, $a\phi = \text{arc AP} = \text{PQ} = r \sin(\phi - \theta) = \sqrt{(r^2 - a^2)}$ ... (2)

and $$\phi = \theta + \cos^{-1}(a/r)$$

$\therefore$ Eliminating ϕ we obtain the polar equation of the curve in the form :

$$a\theta + a\cos^{-1}\frac{a}{r} = \sqrt{(r^2 - a^2)} \quad \ldots \quad (3)$$

If QM be the ordinate of Q, the co-ordinates of Q are given by—

$$x = \text{OM} = a\cos\phi + \text{PQ}\sin\phi$$
$$= a\cos\phi + a\phi\sin\phi$$
$$y = \text{QM} = a\sin\phi - a\phi\cos\phi$$

It is to be noted, however, that PQ is the normal to the involute and if p is the perpendicular on the tangent at Q to the involute, $p = \text{PQ} = a\phi$; and since $\text{OQ}^2 = \text{OP}^2 + \text{PQ}^2$, the pedal equation of the involute is $r^2 = a^2 + p^2$.

Again, if $\text{AQ} = s$, we have $(\partial s/\partial\phi) = \text{PQ} = a\phi$, from which the intrinsic equation is obtained in the form $s = (a\phi^2/2)$.

It will be noticed that the involute of the circle is the locus of the intersection of tangents drawn at the points where any ordinate to OA meets the circle and the corresponding cycloid having its vertex at A.

Again, if the involute of a circle roll on a straight line, the locus of the centre of the circle of which it is the involute is a parabola.

For if (x, y) be the co-ordinates of the centre of the circle referred to the point on the line with which A was originally in contact, then

$$y = p^* \qquad \tan\phi = \frac{\partial x}{\partial y} = \frac{p}{\sqrt{(r^2 - p^2)}} = \frac{y}{a}$$

whence we obtain $\quad y^2 = 2ax.$

* See Basset, *loc. cit.*, p. 209.

283. THE CURVE OF SINES, HARMONIC CURVE, COMPANION TO THE CYCLOID:

A large class of transcendental curves is obtained by taking the ordinate some trigonometrical function of the abscissa. The shape of these curves can be easily derived from their equations.

The curve defined by the equation $y=\sin x$ is called the *curve of sines.*

This curve has positive and constantly increasing ordinates until $x=\frac{1}{2}\pi$, after which the ordinates decrease, until $x=\pi$. At this point the curve crosses the axis of x at an angle of 45°. For values of x between π and 2π, the ordinates are negative, and the curve has a similar portion on the negative side of the axis. The curve, therefore, consists of an infinity of similar portions on alternate sides of the x-axis. There are points of inflexion wherever the curve cuts the x-axis and it lies entirely between the lines $y=\pm 1$.

The curve defined by the equation $y=m \sin x$ is sometimes called the *harmonic curve*, and only differs from the curve of sines in that its ordinates are each m times the corresponding ordinates of the latter.

In a like manner we may consider the graph of the equation $y=\tan x$. Here the ordinates increase regularly from $x=0$ to $x=\frac{1}{2}\pi$, when y becomes infinite and the line $x=\frac{1}{2}\pi$ is an asymptote. For greater values of x from $\frac{1}{2}\pi$ to π, y changes from negative infinity to 0. Thus the curve consists of an infinity of infinite branches, possessing an infinity of asymptotes $x=\frac{1}{2}\pi$, $x=\frac{3}{2}\pi$, etc. There are inflexions at the points $x=0$, $x=\pi$, $x=2\pi$, etc.

Curves represented by the equations $y=\cos x$, $y=\sec x$, etc., can be studied in a similar manner.

The curve known as the *Companion to the cycloid* is generated by producing the ordinates of a circle, not as in the case of the cycloid, until the produced part is equal

to the arc, but until the entire ordinate is equal to the arc. The curve may be represented by the equations

$$x = a\theta, \qquad y = a(1 - \cos\theta)$$

or

$$y - a = a\sin\left(\frac{x}{a} - \frac{\pi}{2}\right)$$

and therefore the locus is the harmonic curve.

284. THE LOGARITHMIC CURVE:

We have so far discussed curves depending on trigonometrical functions. We shall now mention those depending on exponential functions. Such a curve, known as the *logarithmic curve*, is characterised by the property that the abscissa is proportional to the logarithm of the ordinate, and the equation of the curve, therefore, is—

$$x = m\log y \quad \text{or} \quad y = a^x$$

In the simplest form, it may be taken as—

$$x = \log_e y \quad \text{or} \quad y = e^x.$$

When x is negative and very large, the ordinate diminishes without limit, and the x-axis towards $-\infty$ becomes asymptotic. It cuts the axis of y at a distance equal to the unit length, and x and y then increase together to positive infinity. The subtangent of the curve is constant; since

$$\text{subtangent} = \frac{y\partial x}{\partial y} = m.$$

The subnormal is given by—

$$y(\partial y/\partial x) = (y^2/m)$$

The radius of curvature is given by—

$$\rho = \frac{(y^2 + m^2)^{\frac{3}{2}}}{my}$$

285. THE EQUIANGULAR OR LOGARITHMIC SPIRAL :

We shall now consider a class of curves known as *spirals*. When referred to polar co-ordinates, the radius vector is not a periodic function of the angle, but one, which gives an infinity of different values when the angle is increased by multiples of 2π. Consequently, a straight line meets the curve in an infinity of points and the curve is transcendental.

We shall first discuss the *Logarithmic Spiral*, which possesses the characteristic property that the tangent makes a constant angle with the radius vector. This curve was imagined by Descartes and some of its properties were discovered by him. The properties of its reproducing itself in various ways were discovered by James Bernoulli. The curve is sometimes called the *equiangular spiral.*

Let ω be the angle which the radius vector makes with the tangent at any point on the curve. Then we obtain the following differential equation—

$$\frac{r\partial\theta}{\partial r}=\tan\omega=\text{constant}$$

which, on integration, gives the equation of the curve in the form :

$$r=Ce^{\theta\cot\omega} \qquad \ldots \quad (1)$$

Taking the constant C to be unity, the equation can be written in the simple form :

$$r=e^{\theta\cot\omega}$$

In this curve, r increases indefinitely with θ; when $\theta=0$, $r=1$, and diminishes further for negative values of θ, but it does not vanish until θ becomes negative infinity. Hence the curve has an infinity of convolutions before reaching the pole.

286. PROPERTIES OF THE LOGARITHMIC SPIRAL:

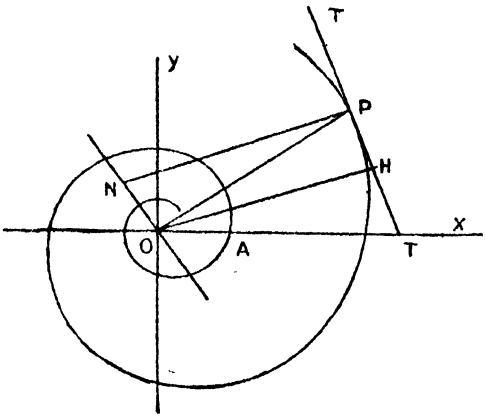
Let O be the pole, PT the tangent at P, OH the perpendicular on PT, ON the polar sub-normal. Let $\angle$ OPT$=\omega$

Then the perpendicular $p=\text{OH}=r\sin\omega$

The polar subnormal$=(\partial r/\partial\theta)=r\cot\omega$

The polar subtangent $=r^2(\partial\theta/\partial r)=r\tan\omega$

The length of normal$=\text{PN}=r\operatorname{cosec}\omega$



The radius of curvature—

$$\rho=r(\partial r/\partial p)=r\operatorname{cosec}\omega=\text{length of the normal.}$$

Thus, the perpendicular, subtangent, subnormal, length of normal and the radius of curvature are all proportional to the radius vector r.

If θ_1 and r_1 be the co-ordinates of N, we have

$$r_1=\text{ON}=r\cot\omega=\cot\omega\, e^{\theta\cot\omega}$$

and $$\theta_1=\angle\text{NOX}=\theta+(\pi/2)$$

Eliminating θ between these, we obtain—

$$r_1=\cot\omega\; e^{\cot\omega\,(\theta_1-\frac{\pi}{2})}=e^{c\theta_1-c\frac{\pi}{2}+\log c},\quad\text{where}\quad c=\cot\omega$$

Putting $\theta_1=\theta_2+\dfrac{\pi}{2}-\dfrac{\log c}{c}$ we obtain $r_1=e^{c\theta_2}$

Hence, *the evolute of the logarithmic spiral is another equal logarithmic spiral referred to the same pole.*

Again, $p = \mathrm{OH} = r \sin \omega = \dfrac{r}{\sqrt{1+\cot^2 \omega}}$.

Also, $\angle \mathrm{POH} = \angle \mathrm{OPN} = \text{a constant} \left(\dfrac{\pi}{2} - \omega \right) \equiv a$

Thus, if (r', θ') be the co-ordinates of H,

$$r' = \frac{r}{\sqrt{1+\cot^2 \omega}} \quad \text{and} \quad \theta' = \theta - a$$

Combining this with the equation of the curve we obtain the equation of the pedal in the form—

$$r' = \frac{e^{\cot \omega (\theta' + a)}}{\sqrt{1+\cot^2 \omega}}$$

which is a logarithmic spiral.

Hence, *the pedal of the logarithmic spiral is also a logarithmic spiral equal to the first.*

Thus, the first positive pedal, and consequently all other pedals are equal equiangular spirals.

For similar considerations, the inverse and the polar reciprocal with regard to the pole are equal spirals. The caustics by reflexion and refraction, the light emanating from the pole, are also logarithmic spirals.

Since $(\partial r / \partial s) = \cos \omega$, we have $r = s \cos \omega$

when the arc is measured from the pole where $r = 0$.

If K be the point where the tangent meets the line NO produced, we have $\mathrm{PK} \cos \omega = r \quad \therefore \quad s = \mathrm{PK}$

Hence, *any arc is equal to the difference of the extreme radii vectores multiplied by the secant of the constant angle.*

It will be easily seen that if the spiral rolls along a fixed line, the locus of the pole and also of the centre of curvature of the point of contact is a straight line.

287. The Parabolic Spiral :

The system of curves represented by the equation

$$(r-a)^2=2pa\theta$$

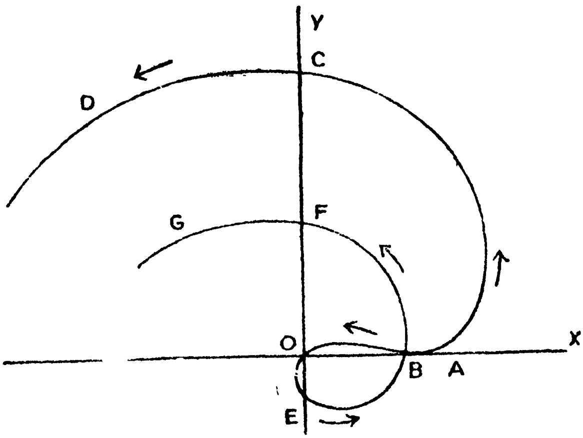
are called Parabolic Spirals.*

If in the equation we put $p=(b^2/a)$, and then make $a=0$, the equation reduces to $r^2=2b^2\theta$, which represents the Spiral of Fermat.

The parabolic spiral consists of two branches, the first ABCD...corresponds to the equation—

$$r=a+\sqrt{2pa\theta}$$

and starts at the point A where $\theta=0$ and $r=OA=a$ and makes an infinite number of turns round the origin, receding indefinitely from the point.



The second branch AOEFG...corresponding to the equation—

$$r=a-\sqrt{2ap\theta}$$

starts at the same point A and gradually approaches towards the point O, until $\theta=a/2p$, and then receds indefinitely from the same point.

* These were studied by James Bernoulli.

The tangent to the curve at the origin O forms an angle equal to $(a/2p)$ with the axis of x, and the axis of x is the tangent to the curve at the point A.

If ϕ denotes the angle which any tangent makes with the radius vector to the point of contact, we have

$$\tan\phi = \frac{r\partial\theta}{\partial r} = \frac{r(r-a)}{ap}$$

and the subtangent $= \dfrac{r^2(r-a)}{ap}$.

In order to determine the double points on the curve, it is necessary to find two values of θ which differ by $(2n+1)\pi$, where n is a positive integer, and the corresponding values of r are equal, but of opposite signs.

At a double point, then, we must have—

$$r = a + \sqrt{2pa\theta}, \qquad -r = a + \sqrt{2pa[\theta+(2n+1)\pi]}$$

or, $$r = a - \sqrt{2pa\theta}, \qquad -r = a - \sqrt{2pa[\theta+(2n+1)\pi]}$$

showing that at the double point two branches of the curve intersect or the same branch crosses itself.

Eliminating r between the above values, we obtain—

$$[4a^2 - 2pa\{2\theta+(2n+1)\pi\}\ ]^2 = 16p^2a^2\theta[\theta + (2n+1)\pi]$$

whence $$\theta = \frac{[2a-p(2n+1)\pi]^2}{8ap}$$

n being a positive integer. This gives the vectorial angle of the double point.

For determining the points of inflexion, we make use of the formula—

$$r^2 - r\frac{\partial^2 r}{\partial\theta^2} + 2\left(\frac{\partial r}{\partial\theta}\right)^2 = 0$$

and obtain—

$$r^2(r-a)^3 + 2p^2a^2(r-a) + p^2a^2r = 0$$

This equation combined with the equation of the curve gives the values of r and θ corresponding to the points of inflexion, and it follows that there is at least one real point of inflexion.

It can be easily seen that the real points of inflexion of the curve are situated in the interior of a circle of radius a with centre at the origin, and the value of r at these points are positive.

The system of curves represented by the equation $r = a\theta^n$ is also called parabolic spirals,—the most remarkable of these are the following:

When $n=1$, it represents the Archimedean Spiral,

$n=-1$, ,, ,, ,, Hyperbolic or Reciprocal Spiral,

$n=-\frac{1}{2}$, ,, ,, ,, Lituus.

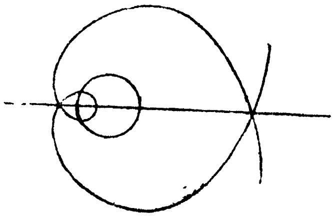
The family of curves represented by this equation is such that the tangent approaches more nearly to being perpendicular to the radius vector, the further the point recedes from the origin, since

$$\tan\phi = \frac{r\partial\theta}{\partial r} = \frac{\theta}{n}$$

$\therefore$ The tangent of the angle which the tangent makes with the radius vector increases as θ increases, but does not actually become infinity until θ becomes infinite.

288. THE SPIRAL OF ARCHIMEDES:

This curve is represented by the equation $r = a\theta$



This is the path described by a point receding uniformly from the origin, while the radius vector on which it travels moves also uniformly round the origin.*

This curve is the locus of the foot of the perpendicular on the tangent to the involute of a circle (§ 282); for the tangent to the locus of Q is perpendicular to PQ and the length of the perpendicular on that tangent from O will be equal to $PQ = a\phi$.

Again, if a circle be drawn with centre at the pole O and radius a, any radius vector of the curve is equal to the arc of this circle measured from the initial line to the point in which the radius vector cuts the circle.

The subtangent $= (r^2 \partial\theta/\partial r) = r\theta$

The subnormal $= (\partial r/\partial\theta) = a.$

The length of the normal $N = \sqrt{a^2 + r^2}$

and $\tan\phi = (r\partial\theta/\partial r) = (r/a).$

The radius of curvature is given by—

$$\rho = \frac{(r^2 + a^2)^{\frac{3}{2}}}{r^2 + 2a^2} = \frac{N^3}{N^2 + a^2}$$

* This curve is due to Canon, who died before he completed his investigations of its properties, which, however, were continued and completed by Archimedes.

289. THE HYPERBOLIC SPIRAL:

The inverse of the Archimedean Spiral is called the *hyperbolic* or the *reciprocal* spiral and is represented by the equation $r\theta=a$.

The curve is the reciprocal polar of the involute of a circle.

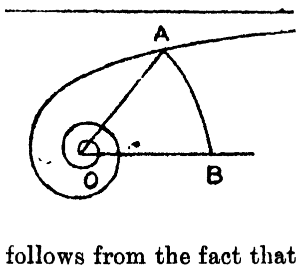
The name *hyperbolic* is derived from the analogy between the form of its equation and that of the hyperbola referred to its asymptotes in Cartesians.

The curve has an asymptote parallel to the initial line at a distance a from the same.

If (x, y) be the Cartesian co-ordinates of any point on the spiral, we have—

$$y=r\sin\theta=\frac{a\sin\theta}{\theta}$$

which, when θ vanishes and r becomes infinite, has the finite values a.



The same also follows from the fact that

$$\tan\phi=\frac{r\partial\theta}{\partial r}=-\theta$$

and the length of the perpendicular drawn from the origin on the tangent $=\dfrac{ar}{\sqrt{a^2+r^2}}$, and this is equal to a, when r becomes infinite.

If a circle be drawn with any radius OA, where A is any point on the curve, and the centre at the origin, the arc of the circle intercepted between the curve and the initial line *i.e.*, the arc AB is of constant length.

The polar subtangent$=r^2(\partial\theta/\partial r)=-a=$constant.

The polar subnormal$=-(a/\theta^2)=-(r^2/a)$

The radius of curvature is given by—

$$\rho=\frac{r(a^2+r^2)^{\frac{3}{2}}}{a^3}=\frac{N^3}{r^2},$$

where N is the length of the normal.

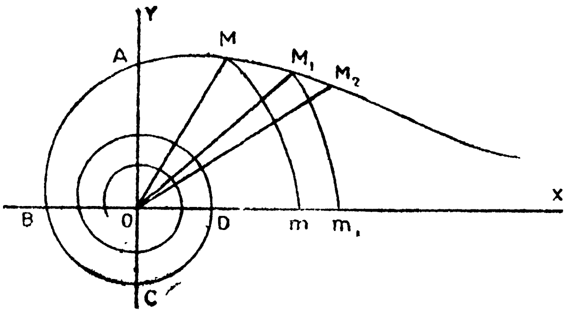
The pedal equation of the curve is

$$(1/p^2)=(1/r^2)+(1/a^2).$$

290 THE LITUUS:

Another spiral curve defined by the equation $r^2\theta=a^2$, called the *lituus*, is of some considerable importance.*

This curve has the initial line for an asymptote.



One remarkable property of the curve is that the area of the circular sector with centre at the origin and bounded by the initial line and the radius vector is constant.

For this area$=\frac{1}{2}r^2\theta=(a^2/2)$

* Cotes—Harmonia mensurarum.

At the points A, B, C, D the values of θ are respectively

$$(\pi/2), \quad \pi, \quad (3/2)\pi, \quad 2\pi, \text{ etc.}$$

and the corresponding values of r are—

$$OA=a\sqrt{\frac{2}{\pi}}, \quad OB=a\sqrt{\frac{2}{2\pi}}, \quad OC=a\sqrt{\frac{2}{3\pi}}, \quad \text{etc.}$$

The polar subtangent at the point (r, θ)

$$=r^2\frac{\partial\theta}{\partial r}=-\frac{2a^2}{r}$$

whence we can construct the tangent to the curve.

The radius of curvature is given by—

$$\rho=\frac{r(4a^4+r^4)^{\frac{3}{2}}}{2a^2(r^4-4a^4)}$$

For determining the points of inflexion we make use of the formula—

$$r\frac{\partial^2 r}{\partial\theta^2}-2\left(\frac{\partial r}{\partial\theta}\right)^2-r^2=0$$

whence $\quad \frac{1}{4}\theta^{-2}-1=0$, and consequently $\theta=\pm\frac{1}{2}$

Corresponding to the value $\theta=-\frac{1}{2}$, r is imaginary. Hence the curve has only one inflexion at the point M_2, where

$$\theta=\tfrac{1}{2}, \quad r=a\sqrt{2}.$$

Besides the spirals described in these articles, there are other curves belonging to this class such as the spiral of Galileo, spiral of Fermat, etc. They are equally interesting, and the students should consult the famous book of Teixeira for detailed accounts of these and other similar curves.

APPENDIX I

NOTES ON SEXTACTIC POINTS OF A CUBIC

In article 21, it has been stated that there are twenty-seven points on a cubic the osculating conic at each of which has six-pointic contact with the curve. Plücker and Steiner gave a method of constructing such points for a curve of the third order. Salmon,* by using an identity, obtained the equation of a conic through five consecutive points of a cubic; but Cayley,† in an elaborate memoir, discussed the conic of five-pointic contact for a general curve of any order, and thence deduced a method of constructing the sextactic points. In another memoir ‡ he pursued the enquiry as to ascertain what conditions must be fulfilled by the co-ordinates of a point in order that the contact may be six-pointic.

Cayley has shown that the sextactic points of a cubic defined by the canonical form—

$$x^3+y^3+z^3+6lxyz=0$$

are determined by the equation—

$$(y^3-z^3)(z^3-x^3)(x^3-y^3)=0$$

combined with the equation of the curve, showing that the twenty-seven sextactic points form nine groups of three

* Salmon—Phil. Transactions (1858), p. 535.

† Cayley—*On the Conic of Five-pointic Contact at any point of a Plane Curve*—Coll. Works., Vol. 4, pp. 207-239.

‡ Cayley—*Sextactic Points of a Plane Curve,*—Coll. Works, Vol. 5 pp. 221-257.

each, such that the three points of a group have for their common tangential one of the nine inflexions.

If ω be a cube root (real or imaginary) of unity, the three sextactic points of one of the groups are given by

$$x-\omega y=0$$

$$x^3+y^3+z^3+6lxyz=0$$

The co-ordinates of the tangential of any one of these points are—

$$x_1:y_1:z_1=x(y^3-z^3):y(z^3-x^3):z(x^3-y^3)$$

$$=\omega y(x^3-z^3):-y(x^3-z^3):0$$

whence $\qquad x_1+\omega y_1=0, \qquad z_1=0;$

i.e., the point (x_1, y_1, z_1) is one of the points of inflexion. This in fact is the construction given by Plücker and Steiner.

The conic of five-pointic contact meets the cubic in a sixth point which is called the point of simple intersection, and it is shown that the point of simple intersection of the cubic and the five-pointic conic is the third point of intersection with the cubic, of the line joining the point of contact with the second tangential of this point.

The construction for the sextactic points follows at once from that for the point of simple intersection. Consider a point having for its tangential a point of inflexion; but a point of inflexion is its own tangential and the second tangential of the point is consequently the point of inflexion. The line* joining the point with the second tangential is,

* Salmon has shown that the line joining the point of five-pointic contact and the point of simple intersection meets the cubic again in the second tangential of the point of contact.

therefore, the tangent at the point, and the point of simple intersection coincides with the point itself, *i.e.*, the point becomes a sextactic point.

Cayley, by using his general formula, obtained the equation of the five-pointic conic for a cubic, which was afterwards simplified by means of Salmon's identity. He gave the equation of the conic for the canonical cubic in the following form—

$$aX^2+bY^2+cZ^2+2fYZ+2gZX+2hXY=0$$

where
$$\begin{aligned} a= & \, x^{10}+10lx^8yz+40l^2x^6y^2z^2 \\ & +(5+120l^3)x^4y^3z^3 \\ & -10lx^2y^4z^4-4l^2y^5z^5 \end{aligned}$$

... ...

$$\begin{aligned} 2f= & -4lx^{10}-40l^2x^8yz+(5-120l^3)x^6y^2z^2 \\ & +40lx^4y^3z^3+8l^2x^2y^4z^4-2y^5z^5 \end{aligned}$$

... ...

(x, y, z) being the co-ordinates of the point.

The investigation for the cubic was, however, extended to a curve of any order in his first memoir referred to above, and thus the problem was completely solved.

Salmon * has given the following method of deducing the equation of the five-pointic conic for a cubic defined by the general equation.

In the equation $U=0$ of a cubic, substitute $x+\lambda x'$, $y+\lambda y'$, $z+\lambda z'$ for x, y, z respectively and the result † becomes—

$$U+3\lambda\Delta U+3\lambda^2\Delta^2U+\lambda^3U'=0$$

* Salmon—H. P. Curves, §§ 236 and 237.

† See Theory of Plane Curves, Vol. I, § 67.

i.e., of the form—

$$U+3\lambda S+3\lambda^2 P+\lambda^3 U'=0 \qquad \dots \quad (1)$$

Writing the result of a similar substitution in the Hessian $H=0$ in the form—

$$H+3\lambda\Sigma+3\lambda^2\Pi+\lambda^3 H'=0 \qquad \dots \quad (2)$$

it is easily verified that

$$3(S\Pi-\Sigma P)=H'U-HU'$$

Hence when (x', y', z') is a point on the cubic U, $U'=0$, and U may be written in the form—

$$S\Pi-\Sigma P=0 \qquad \dots \quad (3)$$

Now, since S touches the cubic, and P is the common tangent, the general equation of a conic touching U at (x', y', z')

is $$S-LP=0,$$

where $L\equiv \alpha x+\beta y+\gamma z$ is an arbitrary line.

By means of the identity (3), the equation of the cubic may be written in the form—

$$\Pi(S-LP)=P(\Sigma-L\Pi)$$

Thus the four points where $S-LP$ meets the cubic again are its intersections with $\Sigma-L\Pi$; and if the latter conic pass through (x', y', z') the former will pass through three consecutive points on the cubic. But putting x', y', z' for

x, y, z we obtain $\Sigma'=\Pi'=H'$, and the condition that $\Sigma-L\Pi$ should pass through (x', y', z') is $L'=1$.

Next, for $S-LP$ to pass through four consecutive points, $\Sigma-L\Pi$ must have P for a tangent at (x', y', z).

But the tangent to $\Sigma-L\Pi$ is—

$$2\Pi-L'\Pi-L\Pi'=0$$

which is the polar of (x', y', z'), and since $L'=1$, and $\Pi'=H'$, this reduces to $\Pi-H'L$, which is to be proportional to P.

$$\therefore \quad L=\theta P+\frac{1}{H'}\Pi$$

Consequently, the general equation of a conic through four consecutive points is—

$$S-\theta P^2-\frac{1}{H'}P\Pi=0 \qquad \dots \quad (4)$$

And $$\Sigma-\theta P\Pi-\frac{1}{H'}\Pi^2=0$$

passes through the two points where the first conic meets the cubic again, since the equation of the cubic is reducible to the form—

$$\Pi(S-\theta P^2-\frac{1}{H'}P\Pi)=P(\Sigma-\theta P\Pi-\frac{1}{H'}\Pi^2) \qquad \dots \quad (5)$$

Since P is a common tangent to the two conics, the expression obtained by adding their equations multiplied by suitable constants will be divisible by P, *i.e.*, we shall require to find the value of μ such that—

$$\mu S+\Sigma-\frac{1}{H'}\Pi^2$$

may represent two right lines, *i.e.*, the discriminant of this must vanish.

The discriminant is found to be—

$$\mu^3 H' + 4\mu^2 \frac{\Theta}{H'}$$

which is to be zero.

If now $\mu = -\frac{4\Theta'}{H'^2}$, we have

$$\mu S + \Sigma - \frac{1}{H'}\Pi^2 = P.M \qquad \dots \quad (6)$$

where M is a linear factor.

The equation of the cubic (5) may be put into the form—

$$(\Pi + \mu P)(S - \theta P - \frac{1}{H'}P\Pi) = P^2\{M - \frac{\mu}{H'}\Pi - \theta(\Pi + \mu P)\}$$

The equation shows that $\Pi + \mu P$ is the tangent at the tangential of the given point on the cubic, and that $M - (\mu/H')\Pi$ passes through the second tangential of the given point.

Next, in order that the conic may pass through five consecutive points, the co-ordinates (x', y', z') must satisfy the equation—

$$M - \frac{\mu}{H'}\Pi - \theta(\Pi + \mu P) = 0 \qquad \dots \quad (7)$$

Differentiating the equation (6) and substituting the co-ordinates x', y', z' for (x, y, z) in the result, we obtain—

$$M' = 2\mu$$

for $$\frac{\partial S'}{\partial x'} = 2\frac{\partial P'}{\partial x'}, \qquad \frac{\partial \Sigma'}{\partial x'} = 2\frac{\partial \Pi'}{\partial x'}.$$

Hence, when (x', y', z') are put for (x, y, z) in (6), we obtain—

$$\mu-\theta H'=0; \qquad \text{but} \quad \mu=-\frac{4\Theta'}{H'^2}$$

$$\therefore \quad \theta=-\frac{4\Theta'}{H'^3}$$

Substituting this value of θ in the equation (3) we obtain the equation of the conic of five-pointic contact in the form—

$$S+\frac{4\Theta'}{H'^3}P^2-\frac{P\Pi}{H'}=0 \qquad .. \quad (8)$$

Prof. Cayley, however, in his memoir referred to, investigated this conic of a curve of order m, and then determined the conditions which must be fulfilled by (x', y', z') in order that the contact may be six-pointic.*

His result is that (x', y', z') must satisfy the equation—

$$(m-2)(12m-27)HJ(U, H, \Phi)$$

$$-3(m-1)HJ'(U, H, \Phi)$$

$$+40(m-2)^2J(U, H, \Theta)=0$$

where, by $J(U, H, \Phi)$ is meant the Jacobian of these three functions, and by J' is meant that, in taking the Jacobian, Φ is to be differentiated on the supposition that the second differential co-efficients of H, which enter into the expression for Φ, are constant. The equation represents a curve of order $12m-27$ whose intersection with U determines $m(12m-27)$ sextactic points.

* Cayley—Coll. Works, Vol. 5, p. 221.

William Spottiswoode* gave a method of determining the sextactic points by applying a method discussed in his memoir.† He obtained for the equation of condition, in place of Cayley's condition, one of order $(18m-36)$, which contained other extraneous factors. Then removing these factors he obtained the sextactic points as the intersections of a curve of order $12m-27$ with the given curve of order m.

* W. Spottiswoode—*On the Sextactic Points of a Plane Curve*—Phil. Trans. (1865), pp. 653-669.

† W. Spottiswoode—*On the Contact of Curves*—Phil. Trans., (1867), p. 41.

APPENDIX II

Notes on the Bicircular Quartic

Dr. Casey, in his remarkable memoir on bicircular quartics,* gave a number of interesting methods of generation of bicircular quartics, and thus obtained various properties of the curve. His results form mainly the contents of these notes.

The most general equation of a bicircular quartic may be written in the form—

$$U \equiv (a,\ b,\ c,\ f,\ g,\ h,)(\alpha,\ \beta,\ \gamma)^2 = 0 \qquad \ldots \quad (1)$$

where $\alpha=0$, $\beta=0$, $\gamma=0$ are the equations of three given circles. By analogy to the trilinear system, the circles α, β, γ are called the *circles of reference.*

The equation

$$x\alpha + y\beta + z\gamma = 0 \qquad \ldots \quad (2)$$

where x, y, z are variable multiples, represents a circle. This will touch the locus (1), provided

$$\begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & 0 \end{vmatrix} = 0 \qquad \ldots \quad (3)$$

that is to say, the locus (1) is the envelope of the circle (2), provided the condition (3) is satisfied.

* Casey—Proc. of the Royal Irish Academy, Vol. 10 (1869), p. 44.

Now, if A, B, C be the centres of the three circles α, β, γ, it is evident that the centre O of the circle (2) is the mean centre of the points A, B, C. Also the circle (1) is co-orthogonal with the three circles α, β, γ.

Let $(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3)$

be the co-ordinates of A, B, C, respectively, and (X, Y, Z) those of the point O.

Then,
$$X = x.x_1 + y.x_2 + z.x_3$$
$$Y = x.y_1 + y.y_2 + z.y_3$$
$$Z = x.z_1 + y.z_2 + z.z_3$$

Therefore, X, Y, Z are proportional to x, y, z, when they are regarded as areal co-ordinates of O, referred to ABC. Thus the locus of (X, Y, Z) is given by (3), and we obtain the theorem, as was otherwise obtained in § 168 :—

If $(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2 = 0$ *be the equation of a bicircular quartic, this quartic is the envelope of a variable circle, co-orthogonal with* α, β, γ, *whose centre moves along the conic whose equation in areal co-ordinates is the determinant* (3), *referred to the circles* α, β, γ.

The conic (3) is the focal conic of the bicircular quartic.

The equation $(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2 = 0$ can be thrown into the form $SV_1 = kS_1V$, where k is a constant, and S, V, S_1, V_1 are four circles. Now, $SV_1 = kS_1V$ is the result of eliminating λ between the equations—

$$S + k\lambda S_1 = 0,$$

$$V + \lambda V_1 = 0.$$

Now, by varying λ, $S + k\lambda S_1$ gives a pencil of circles coaxial with S, S_1, and $V + \lambda V_1$ is a homographic pencil coaxial with V, V_1.

Hence the following theorem is easily deduced:

A bicircular quartic may be described by the intersection of the homologous circles of two homographic pencils of circles.

From this, an equally simple method of describing a bicircular quartic is easily derived:—

Consider two systems of coaxial circles through the points A, B and C, D respectively.

Institute a homographic relation between the circles of the two systems such that, to the circle with centre O corresponds the circle with centre O′; the centres E and E of the corresponding members are then determined so that $\angle EAO = \angle E'DO'$. Let these homologous circles intersect in the points P, P′. Then the locus of P and P′ is a bicircular quartic.

$$\text{Now,} \quad \angle APB - \angle CPD = \tfrac{1}{2}\angle AEB - \tfrac{1}{2}\angle CE'D$$

$$= \tfrac{1}{2}\angle AOB - \tfrac{1}{2}\angle CO'D$$

$$= \text{constant}.$$

Thus, the bicircular quartic may be regarded as the locus of the common vertex P of two triangles APB, CPD, such that the difference of their vertical angles is constant.

Hence we obtain the following theorem:

The locus of the common vertex of two triangles whose bases are given and whose vertical angles have a given difference is a bicircular quartic.

Similarly, when the sum of the vertical angles is given, the locus is a bicircular quartic.

Classification of bicircular quartics:

Dr. Casey takes as the basis of his classification the species of the focal conic, which may be (1) an ellipse or a hyperbola, (2) a circle, (3) a parabola. Curves corresponding to each of those divisions have definite and marked

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distinctions, which have been discussed in some details :—

(1) The case when the focal conic is an ellipse or a hyperbola has been considered in the body of the text.

(2) When the focal conic is a circle, the bicircular quartic becomes a Cartesian oval.

We have seen that the foci of the focal conic are the double foci of the bicircular quartic. Now, when the two foci coincide, the focal conic becomes a circle, and the tangents to the quartic at the circular points at infinity coincide, or, in other words, these points become cusps and the quartic becomes a Cartesian oval.

Hence, *a Cartesian oval is a bicircular quartic whose focal conics are circles.*

In the case of the Cartesian oval, the four confocal (focal) conics become three concentric circles and a right line through their centre.*

The centres of inversion of a Cartesian oval are the foci of the curve.

Let the equations of the focal circle J and the circle of inversion F be—

$$(x+a)^2+y^2=r^2 \quad \text{and} \quad x^2+y^2=r^2$$

respectively. Then the perpendicular OT let fall from O, the centre of F, on a tangent to J is equal to $r-a\cos\theta$, where θ is the angle which OT makes with the axis of x. Then P, P′, two points on OT, such that

$$OT^2-TP^2=OT^2-TP'^2=k^2,$$

are points on the Cartesian oval.

* *Cf.* § 222.

If now we denote OP by ρ, we obtain—

$$k^2+\rho^2=\mathrm{OT}^2-\mathrm{TP}^2+(\mathrm{OT}+\mathrm{TP})^2=2\mathrm{OT}.\mathrm{OP}.$$

i.e. $$2(r-a\cos\theta)\rho=k^2+\rho^2,$$

which may be written in the form—

$2r\rho=\mathrm{C}$, where C is a circle concentric with F

$$\therefore\quad 4r^2\rho^2=\mathrm{C}^2,\quad \text{or,}\quad 4r^2(x^2+y^2)=\mathrm{C}^2.$$

Hence, the circular lines $x\pm iy=0$ are tangents to the curve, so that the centre of the circle of inversion F is a focus.

The equation of a Cartesian oval shows that it is the envelope of the circle—

$$(x^2+y^2)+\mu\mathrm{C}+\mu^2r^2=0.$$

This circle has double contact with the oval. When $\mu=-1$, the circle becomes a right line, which shows that the oval has a bitangent. The expression for the radius of the circle is of the third degree in μ, and consequently, there are three values of μ which reduce it to a point-circle. Hence there are three collinear *single* foci of the curve (§ 219).

(3) When the focal conic is a parabola, the quartic reduces to a circular cubic and the line at infinity. Since the focal conic is a parabola, some of the methods given before require modifications; and, moreover, some methods are applicable to circular cubics, which have no analogues in bicircular quartics.

Thus, *the locus of the common vertex of two triangles whose bases are given, and whose vertical angles are equal, is a circular cubic.*

Dr. Casey thus deduced a number of interesting properties of circular cubics.

(1) *The osculating circles at the centres of inversion are the inverses of the asymptote with respect to the circles of inversion corresponding to these points.*

For, the inverse of the asymptote *w.r.t.* a centre of inversion is a circle through the origin and the two consecutive points at infinity on the asymptote invert into two consecutive points at the origin. Hence the asymptote inverts into the osculating circle.

(2) *The nine-points circle of the triangle formed by any three of the four centres of inversion of a circular cubic passes through the point where it meets the asymptote* (§ 205).

Let D, E, F, be the feet of the perpendiculars of the triangle. Then the nine-points circle passes through these points.

Suppose the nine-points circle meets the cubic in a fourth point V, and G′ is the point where EF meets the cubic again; then DO′ is parallel to the asymptote,* and if DV meets the cubic in O″, then O′O″ is parallel to the asymptote. Hence O″ coincides with D and DV is a tangent at D; consequently V is the point where it intersects the asymptote.

(3) *The tangents at D, E, F intersect at a point on the nine-points circle, where the asymptote meets the cubic.*

From this was immediately deduced the following theorem :—

(4) *The circle described through any three of the centres of inversion of a circular cubic meets the curve again where the cubic is met by the osculating circle at the fourth centre of inversion.*

* For A, B, C, O being the four centres of inversion, A, F, E, O lie on the cubic as well as on a circle, and O′D is parallel to the asymptote. In fact, if the sides DE, EF, FD of the triangle DEF meet the cubic again in F′, D′, E′ respectively, then DD′, EE′, FF′ are parallel to the asymptote (§ 202).

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